Time-Inconsistent Optimal Control Problems and the Equilibrium HJB Equation*

Jiongmin Yong

Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

March 2, 2013

Abstract

A general time-inconsistent optimal control problem is considered for stochastic differential equations with deterministic coefficients. Under suitable conditions, a Hamilton-Jacobi-Bellman type equation is derived for the equilibrium value function of the problem. Well-posedness and some properties of such an equation is studied, and time-consistent equilibrium strategies are constructed. As special cases, the linear-quadratic problem and a generalized Merton's portfolio problem are investigated.

Keywords. time-inconsistent optimal control problem, equilibrium value function, equilibrium Hamilton-Jacobi-Bellman equation, forward-backward stochastic differential equation,

AMS Mathematics subject classification (2010). 93E20, 49L20, 49N10, 49N70, 35Q93.

1 Introduction

Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space on which a d-dimensional standard Brownian motion $W(\cdot)$ is defined, whose natural filtration is $\mathbb{F}=\{\mathcal{F}_t\}_{t\ge0}$ (augmented by all the $\mathbb{P}$-null sets). Let $T>0$. For any $t\in[0,T)$, we consider the following controlled stochastic different equation (SDE, for short):

$$\begin{cases} dX(s)=b(s,X(s),u(s))ds+\sigma(s,X(s),u(s))dW(s), \qquad s\in[t,T],\\ X(t)=x,\end{cases}\tag{1.1}$$

where $b:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^n$ and $\sigma:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^{n\times d}$ are suitable deterministic maps with U being a metric space. In the above, $(t,x)\in[0,T)\times\mathbb{R}^n$ is called an *initial pair*, $u:[t,T]\times\Omega\to U$ is called a *control process* and $X:[0,T]\times\Omega\to\mathbb{R}^n$ is called a *state process*. We define the set of all *admissible controls* by the following:

$$\mathcal{U}[t,T]=\Big\{u:[t,T]\times\Omega\to U\bigm| u(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable}\Big\}.\tag{1.2}$$

Under some mild conditions, for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]$, (1.1) admits a unique solution $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))$. To measure the performance of the control process $u(\cdot)\in\mathcal{U}[t,T]$, we introduce the following cost functional

$$J^0(t,x;u(\cdot))=\mathbb{E}_t\Big[\int_t^T e^{-\delta(s-t)}g^0(s,X(s),u(s))ds+e^{-\delta(T-t)}h^0(X(T))\Big],\tag{1.3}$$

*This work is supported in part by NSF Grant DMS-1007514. Part of the work was done while the author was visiting Department of Applied Mathematics, Polytechnic University, Hong Kong. The author would like to thank Professors X. Li and J. Huang for their hospitality.

with some constant $\delta \geq 0$ (called the *discount rate*), some deterministic maps $g^0 : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $h^0 : \mathbb{R}^n \to \mathbb{R}$, and $\mathbb{E}_t[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\,\mathcal{F}_t]$. On the right hand side of (1.3), the first term is referred to as a *running cost* and the second term is referred to as a *terminal cost*. We can pose the following optimal control problem.

Problem (C). For any $(t,x) \in [0,T) \times \mathbb{R}^n$, find a $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that

$$J^0(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J^0(t,x;u(\cdot)) \equiv V^0(t,x). \tag{1.4}$$

Any $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ satisfying (1.4) is called an *optimal control* of Problem (C) for the initial pair (t,x), the corresponding state process $\bar{X}(\cdot) \equiv X(\cdot\,;t,x,\bar{u}(\cdot))$ and the pair $(\bar{X}(\cdot),\bar{u}(\cdot))$ are called the corresponding *optimal state process* and *optimal pair*, respectively. The function $V^0(\cdot\,,\cdot)$ defined by (1.4) is called the *value function* of Problem (C).

For Problem (C), we have the following Bellman's principle of optimality ([8, 29]): For any $\tau \in [t,T]$,

$$V^0(t,x) = \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \mathbb{E}_t\Big[\int_t^\tau e^{-\delta(s-t)} g^0(s,X(s),u(s))ds + e^{-\delta(\tau-t)} V^0\big(\tau, X(\tau;t,x,u(\cdot))\big)\Big], \tag{1.5}$$

where $\mathcal{U}[t,\tau]$ is defined similar to that of $\mathcal{U}[t,T]$ replacing $[t,T]$ by $[t,\tau]$ (see (1.2)). Now, if $(\bar{X}(\cdot),\bar{u}(\cdot))$ is an optimal pair of Problem (C) for the initial pair $(t,x) \in [0,T) \times \mathbb{R}^n$, then from (1.5), we obtain

$$\begin{aligned}
&V^0(t,x) = J^0(t,x;\bar{u}(\cdot)) \\
&= \mathbb{E}_t\Big[\int_t^\tau e^{-\delta(s-t)} g^0(s,\bar{X}(s),\bar{u}(s))ds + e^{-\delta(\tau-t)} J^0\big(\tau, \bar{X}(\tau;t,x,u(\cdot)); \bar{u}(\cdot)\big|_{[\tau,T]}\big)\Big] \\
&\geq \mathbb{E}_t\Big[\int_t^\tau e^{-\delta(s-t)} g^0(s,\bar{X}(s),\bar{u}(s))ds + e^{-\delta(\tau-t)} V^0\big(\tau, \bar{X}(\tau;t,x,\bar{u}(\cdot))\big)\Big] \\
&\geq \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \mathbb{E}_t\Big[\int_t^\tau e^{-\delta(s-t)} g^0(s,X(s),u(s))ds + e^{-\delta(\tau-t)} V^0\big(\tau, X(\tau;t,x,u(\cdot))\big)\Big] = V^0(t,x).
\end{aligned} \tag{1.6}$$

Thus, one must have

$$\mathbb{E}_t\Big[J^0\big(\tau,\bar{X}(\tau);u(\cdot)\big|_{[\tau,T]}\big) - V^0(\tau,\bar{X}(\tau))\Big] = 0, \qquad \text{a.s.}$$

Since

$$J^0\big(\tau,\bar{X}(\tau);u(\cdot)\big|_{[\tau,T]}\big) - V^0(\tau,\bar{X}(\tau)) \geq 0, \qquad \text{a.s. },$$

it follows that

$$J^0\big(\tau,\bar{X}(\tau);u(\cdot)\big|_{[\tau,T]}\big) = V^0(\tau,\bar{X}(\tau)) = \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J^0\big(\tau,\bar{X}(\tau);u(\cdot)\big), \qquad \text{a.s.} \tag{1.7}$$

This means that the restriction $\bar{u}(\cdot)\big|_{[\tau,T]} \in \mathcal{U}[\tau,T]$ of an optimal control $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ for the initial pair (t,x) on any later time interval $[\tau,T]$ is optimal for the initial pair $\big(\tau,\bar{X}(\tau;t,x,\bar{u}(\cdot))\big)$. Such a phenomenon is called the *time-consistency* of Problem (C).

The advantage of the time-consistency is that for any given initial pair (t,x), if an optimal control $\bar{u}(\cdot)$ can be constructed for that (initial pair), then it will stay optimal hereafter (for the later initial pair along the optimal trajectory). This is very ideal. However, in reality, the time-consistency could be lost. An interesting situation that we are going to discuss in this paper is what we call the *general discounting* situation which includes the so-called *non-exponential discounting*, or *hyperbolic discounting* situations. This amounts to saying the following: Due to the possible subjectivity of people's preferences, the discount factors $e^{-\delta(s-t)}$ and $e^{-\delta(T-t)}$ appeared in (1.3) might be replaced by some general functions $\lambda(s,t)$ and $\nu(T,t)$, or more generally, we will consider the following cost functional:

$$J(t,x;u(\cdot)) = \mathbb{E}_t\Big[\int_t^T g(t,s,X(s),u(s))ds + h(t,X(T))\Big], \tag{1.8}$$

where the maps $g(\cdot)$ and $h(\cdot)$ explicitly depend on the initial time t in some general way. The optimal control problem associated with (1.1) and (1.8), called Problem (N), will not be time-consistent, or *time-inconsistent*, in general, meaning that a restriction of an optimal control for a specific initial pair on a later time interval might not be optimal for that corresponding initial pair. Some concrete examples will be presented in the next section.

In recent years, time-inconsistent optimal control problems have attracted a number of researchers. See [5, 6, 2, 27, 19, 28, 7, 3, 13] and references cited therein.

The purpose of this paper is to obtain time-consistent optimal controls (which should be more properly called *equilibrium control*) for Problem (N) mentioned above. Let us now briefly describe our approach. Inspired by [27, 28], we introduce a sequence of multi-person hierarchical differential games as follows. For any $N > 1$, let Π be a partition of the time interval $[0, T]$ defined by

$$\Pi : 0 = t_1 < t_2 < \cdots < t_N = T,$$

with the mesh size $\|\Pi\|$ given by

$$\|\Pi\| = \max_{1 \le k \le N} (t_k - t_{k-1}).$$

The differential game, denoted by Problem (G^Π), associated with partition Π consists of N players. The k-th player controls the system on $[t_{k-1}, t_k)$ by taking his/her control $u^k(\cdot) \in \mathcal{U}[t_{k-1}, t_k]$. The cost functional is constructed in a sophisticated way, by using some techniques of forward-backward stochastic differential equations (FBSDEs, for short) found in [16, 17]. The interaction among the players are as follows: (i) The terminal pair $(t_k, X(t_k))$ of Player k is the initial pair of Player $(k + 1)$; (ii) All the player know that each player tries to find an optimal control for his/her own problem; and (iii) Each player will discount the future costs in his/her own way, regardless of the fact that the later players will control the system. Under certain conditions, each player will have an optimal control, denoted by $\bar{u}^k(\cdot) \in \mathcal{U}[t_{k-1}, t_k]$, for his/her own problem, as well as his/her own value function $V^k(\cdot\,,\cdot)$ defined on $[t_{k-1}, t_k] \times \mathbb{R}^n$. Define

$$\bar{u}^\Pi(t) = \sum_{k=1}^{N} \bar{u}^k(t) I_{[t_{k-1}, t_k)}(t), \qquad t \in [0, T), \tag{1.9}$$

and

$$V^\Pi(t, x) = \sum_{k=1}^{N} V^k(t, x) I_{[t_{k-1}, t_k)}(t), \qquad (t, x) \in [0, T) \times \mathbb{R}^n. \tag{1.10}$$

We may call $\bar{u}^\Pi(\cdot)$ and $V^\Pi(\cdot\,,\cdot)$ the *Nash equilibrium control* and *Nash equilibrium value function* of Problem (G^Π), respectively. When the following limits

$$\begin{cases} \lim\limits_{\|\Pi\| \to 0} \bar{u}^\Pi(t) = \bar{u}(t), & t \in [0, T], \\ \lim\limits_{\|\Pi\| \to 0} V^\Pi(t, x) = V(t, x), & (t, x) \in [0, T] \times \mathbb{R}^n, \end{cases} \tag{1.11}$$

exist for some $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ and $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, we call them a *time-consistent equilibrium control* and a *time-consistent equilibrium value function* of Problem (N), respectively. As a major contribution of this paper, we have derived the *equilibrium Hamilton-Jacobi-Bellman equation* which can be used to characterize the equilibrium value function $V(\cdot\,,\cdot)$, and it recovers the result for time-inconsistent deterministic linear-quadratic problem presented in [27]. The well-posedness of such an HJB equation will be established for the case that the diffusion of the state equation does not contain the control. The general case is open at the moment, and we expect to present some more complete results in our future publications. As important and interesting special cases, we will construct equilibrium controls for stochastic LQ problem with general discounting and for generalized Merton's portfolio problem.

We refer the readers to [23, 22, 10, 20, 24, 15, 14, 21, 4, 12, 11, 1], for some relevant results.

2 Two Examples of Time-Inconsistent Optimal Control Problems

In this section, we present two interesting examples of optimal control problems which are time-inconsistent.

Example 2.1. Consider a one-dimensional controlled linear SDE:

$$\begin{cases} dX(s) = u(s)ds + \sigma X(s)dW(s), \qquad s \in [t,T], \\ X(t) = x, \end{cases} \tag{2.1}$$

with cost functional

$$J(t,x;u(\cdot)) = \mathbb{E}_t\Big[\int_t^T |u(s)|^2 ds + g(t)|X(T)|^2\Big], \tag{2.2}$$

where $\sigma > 0$ is a constant and $g : [0,T] \to (0,\infty)$ is a deterministic non-constant, continuous function. For a fixed initial pair $(t,x) \in [0,T) \times \mathbb{R}$, the LQ problem associated with the above (2.1)–(2.2) is a standard one. We can show that (see Appendix) the problem admits a unique optimal pair $(\bar X(\cdot), \bar u(\cdot))$ given by

$$\begin{cases} \bar X(s) = \dfrac{\sigma^2 + g(t)(e^{\sigma^2(T-s)}-1)}{\sigma^2 + g(t)(e^{\sigma^2(T-t)}-1)}\, e^{-\frac{\sigma^2}{2}(s-t)+\sigma[W(s)-W(t)]}x, \\ \bar u(s) = -\dfrac{\sigma^2 g(t) e^{\sigma^2(T-s)}}{\sigma^2 + g(t)(e^{\sigma^2(T-t)}-1)}\, e^{-\frac{\sigma^2}{2}(s-t)+\sigma[W(s)-W(t)]}x, \end{cases} \qquad s \in [t,T]. \tag{2.3}$$

Now, let $\tau \in (t,T)$. We consider the corresponding LQ problem starting from the initial pair $(\tau, \bar X(\tau))$. Then the optimal pair, denoted by $(\widehat X(\cdot), \widehat u(\cdot))$, is given by

$$\begin{cases} \widehat X(s) = \dfrac{\sigma^2 + g(\tau)(e^{\sigma^2(T-s)}-1)}{\sigma^2 + g(\tau)(e^{\sigma^2(T-\tau)}-1)}\, e^{-\frac{\sigma^2}{2}(s-\tau)+\sigma[W(s)-W(\tau)]}\bar X(\tau) \\ \qquad = \dfrac{\sigma^2 + g(\tau)(e^{\sigma^2(T-s)}-1)}{\sigma^2 + g(t)(e^{\sigma^2(T-s)}-1)} \cdot \dfrac{\sigma^2 + g(t)(e^{\sigma^2(T-\tau)}-1)}{\sigma^2 + g(\tau)(e^{\sigma^2(T-\tau)}-1)} \bar X(s), \\ \widehat u(s) = -\dfrac{\sigma^2 g(\tau) e^{\sigma^2(T-s)}}{\sigma^2 + g(\tau)(e^{\sigma^2(T-\tau)}-1)}\, e^{-\frac{\sigma^2}{2}(s-\tau)+\sigma[W(s)-W(\tau)]}\bar X(\tau) \\ \qquad = \dfrac{g(\tau)}{g(t)} \cdot \dfrac{\sigma^2 + g(t)(e^{\sigma^2(T-\tau)}-1)}{\sigma^2 + g(\tau)(e^{\sigma^2(T-\tau)}-1)}\, \bar u(s), \end{cases} \qquad s \in [\tau,T]. \tag{2.4}$$

Hence, in general

$$\widehat X(s) = \bar X(s), \quad \widehat u(s) = \bar u(s), \qquad s \in [\tau,T],$$

may fail. In fact, we have the following comparison of the cost functional values corresponding to $\bar u(\cdot)\big|_{[\tau,T]}$ and $\widehat u(\cdot)$, with the same initial pair $(\tau, \bar X(\tau))$:

$$\begin{aligned} &J\big(\tau, \bar X(\tau); \bar u(\cdot)\big|_{[\tau,T]}\big) - J(\tau, \bar X(\tau); \widehat u(\cdot)) = \frac{\sigma^4 (e^{\sigma^2(T-\tau)}-1) e^{\sigma^2(T-\tau)} [g(t)-g(\tau)]^2 |\bar X(\tau)|^2}{[\sigma^2 + g(t)(e^{\sigma^2(T-\tau)}-1)]^2 [\sigma^2 + g(\tau)(e^{\sigma^2(T-\tau)}-1)]} \\ &= \frac{\sigma^4 (e^{\sigma^2(T-\tau)}-1)[g(t)-g(\tau)]^2 e^{\sigma^2(T-\tau)-\sigma^2(\tau-t)+2\sigma[W(\tau)-W(t)]} x^2}{[\sigma^2 + g(t)(e^{\sigma^2(T-t)}-1)]^2 [\sigma^2 + g(\tau)(e^{\sigma^2(T-\tau)}-1)]} > 0, \end{aligned}$$

unless $g(\tau) = g(t)$ or $x = 0$. Therefore, the LQ problem associated with (2.1)–(2.2) is time-inconsistent.

Note that by sending $\sigma^2 \to 0$, using l'Hôpital's rule, we recover the example presented in [27, 28] for a deterministic LQ problem.

Example 2.2. (Merton's portfolio problem) We consider a one-dimensional controlled SDE:

$$\begin{cases} dX(s) = \big[rX(s) + (\mu - r)u(s) - c(s)\big]ds + \sigma u(s) dW(s), \qquad s \in [t,T], \\ X(t) = x, \end{cases} \tag{2.5}$$

where $(u(\cdot),c(\cdot))$ is the control process with $u(\cdot)$ being the dollar amount invested in the stock (which could be positive and negative), $c(\cdot)$ is the consumption rate process (which has to be non-negative), and the solution $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot),c(\cdot))$ of (2.5) corresponding to $(t,x,u(\cdot),c(\cdot))$ is the wealth process, which is required to be non-negative. In the above, $r>0$ is the interest rate for the bank account, $\mu>r$ and σ are the appreciation rate and volatility of the stock, respectively. Let

$$\begin{aligned}&\mathcal{U}[t,T]=\Big\{u:[t,T]\times\Omega\to\mathbb{R}\bigm| u(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable, }\mathbb{E}\Big[\int_t^T|u(s)|^2ds\Big]^{\frac{1}{2}}<\infty\Big\},\\&\mathcal{C}[t,T]=\Big\{c:[t,T]\times\Omega\to[0,\infty)\bigm| c(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable, }\mathbb{E}\int_t^T c(s)ds<\infty\Big\}.\end{aligned}$$

The payoff functional is given by the following:

$$J(t,x;u(\cdot),c(\cdot))=\mathbb{E}_t\Big[\int_t^T\nu(t,s)c(s)^\beta ds+\rho(t)X(T)^\beta\Big],\tag{2.6}$$

where $\nu(\cdot\,,\cdot)$ and $\rho(\cdot)$ are given positive-valued functions, and $\beta\in(0,1)$. As a convention, we define

$$x^\beta=-\infty,\qquad x<0.\tag{2.7}$$

The following is referred to as a *generalized Merton's portfolio problem*:

Problem (M). For any given $(t,x)\in[0,T)\times\mathbb{R}$, find a pair $(\bar u(\cdot),\bar c(\cdot))\in\mathcal{U}[t,T]\times\mathcal{C}[t,T]$ such that

$$J(t,x;\bar u(\cdot),\bar c(\cdot))=\sup_{(u(\cdot),c(\cdot))\in\mathcal{U}[t,T]\times\mathcal{C}[t,T]}J(t,x;u(\cdot),c(\cdot)).\tag{2.8}$$

When

$$\nu(t,s)=e^{-\delta(s-t)},\quad \rho(t)=e^{-\delta(T-t)},\qquad 0\le t\le s\le T,\tag{2.9}$$

the above problem is called a *classical Merton's portfolio problem* (which is time-consistent). We now look at the above general case. For a fixed $t\in[0,T)$, Problem (M) can be regarded as a standard optimal control problem. Therefore, we can treat it in a standard way. We can show that (see Appendix) the problem admits a unique optimal control $(\bar u^t(\cdot),\bar c^t(\cdot))$ for the initial pair (t,x), which is given by the following:

$$\begin{cases}\bar u^t(s)=\dfrac{(\mu-r)\bar X^t(s)}{\sigma^2(1-\beta)},\\[2mm] \bar c^t(s)=\dfrac{\nu(t,s)^{\frac{1}{1-\beta}}\bar X^t(s)}{e^{\frac{\lambda}{1-\beta}(T-s)}\rho(t)^{\frac{1}{1-\beta}}+\int_s^Te^{\frac{\lambda}{1-\beta}(\tau-s)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau},\end{cases}\qquad s\in[t,T],\tag{2.10}$$

with

$$\lambda=\frac{[2r\sigma^2(1-\beta)+(\mu-r)^2]\beta}{2\sigma^2(1-\beta)},\tag{2.11}$$

and

$$\begin{aligned}V^t(t,x)&\equiv\sup_{(u(\cdot),c(\cdot))\in\mathcal{U}[t,T]\times\mathcal{C}[t,T]}J(t,x;u(\cdot),c(\cdot))\\&=\Big[e^{\frac{\lambda}{1-\beta}(T-t)}\rho(t)^{\frac{1}{1-\beta}}+\int_t^Te^{\frac{\lambda}{1-\beta}(\tau-t)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau\Big]^{1-\beta}x^\beta,\qquad (t,x)\in[0,T]\times[0,\infty).\end{aligned}\tag{2.12}$$

Now, let $\bar t\in(t,T)$, and consider Problem (M) starting from the initial pair $(\bar t,\bar X^t(\bar t))$. Then

$$V^{\bar t}(\bar t,\bar X^t(\bar t))=\Big[e^{\frac{\lambda}{1-\beta}(T-\bar t)}\rho(\bar t)^{\frac{1}{1-\beta}}+\int_{\bar t}^Te^{\frac{\lambda}{1-\beta}(\tau-\bar t)}\nu(\bar t,\tau)^{\frac{1}{1-\beta}}d\tau\Big]^{1-\beta}\bar X^t(\bar t)^\beta.$$

We can show that (see Appendix) if $\bar t\in(t,T)$ is such that

$$\int_{\bar t}^T\Big[\frac{e^{\lambda\tau}\nu(t,\tau)}{\rho(t)}\Big]^{\frac{1}{1-\beta}}d\tau\neq\int_{\bar t}^T\Big[\frac{e^{\lambda\tau}\nu(\bar t,\tau)}{\rho(\bar t)}\Big]^{\frac{1}{1-\beta}}d\tau,\tag{2.13}$$

then

$$J\big(\bar t,\bar X^t(\bar t);\bar u(\cdot)\big|_{[\bar t,T]},\bar c(\cdot)\big|_{[\bar t,T]}\big)<V^{\bar t}(\bar t,\bar X^t(\bar t))=\sup_{(u(\cdot),c(\cdot))\in\mathcal U[\bar t,T]\times\mathcal C[\bar t,T]}J(\bar t,\bar X^t(\bar t);u(\cdot),c(\cdot)),\tag{2.14}$$

which means that the restriction $\big(\bar u(\cdot)\big|_{[\bar t,T]},\bar c(\cdot)\big|_{[\bar t,T]}\big)$ of optimal control $(\bar u(\cdot),\bar c(\cdot))$ for the initial pair (t,x) on $[\bar t,T]$ is not optimal for the initial pair $(\bar t,\bar X^t(\bar t))$ if (2.13) holds.

Interestingly, in the case that (2.9) holds, one has

$$\frac{\nu(t,\tau)}{\rho(t)}=\frac{e^{-\delta(\tau-t)}}{e^{-\delta(T-t)}}=e^{\delta(T-\tau)}.$$

Thus, in this case, (2.13) does not hold. As a matter of fact, the problem is time-consistent and (2.13) should not be true.

3 Some Preliminaries

For convenience, let us rewrite the state equation and the cost functional below.

$$\begin{cases}dX(s)=b(s,X(s),u(s))ds+\sigma(s,X(s),u(s))dW(s),\qquad s\in[t,T],\\ X(t)=x,\end{cases}\tag{3.1}$$

and

$$J(t,x;u(\cdot))=\mathbb E_t\Big[\int_t^T g(t,s,X(s),u(s))ds+h(t,X(T))\Big].\tag{3.2}$$

Clearly, our cost functional covers the non-exponential/hyperbolic discounting situations. By comparing the above state equation and cost functional, it seems that we may consider a little more general state equation of the following form:

$$\begin{cases}dX(s)=b(t,x,s,X(s),u(s))ds+\sigma(t,x,s,X(s),u(s))dW(s),\qquad s\in[t,T],\\ X(t)=x.\end{cases}\tag{3.3}$$

However, according to [28] (see also [27]), we know that when an equilibrium pair (see below for definition) is constructed, the eventual effective state equation will take the following form:

$$\begin{cases}dX(s)=b(s,X(s),s,X(s),u(s))ds+\sigma(s,X(s),s,X(s),u(s))dW(s),\qquad s\in[t,T],\\ X(t)=x,\end{cases}\tag{3.4}$$

Therefore, it suffices to consider the state equation of form (3.1).

In what follows, we let $T>0$ be a fixed time horizon, and $U\subseteq\mathbb R^m$ be a closed subset, which could be either bounded or unbounded (it is allowed that $U=\mathbb R^m$). We will use $K>0$ as a generic constant which can be different from line to line. Let $\mathcal S^n$ be the set of all $(n\times n)$ symmetric real matrices. Denote

$$D[0,T]=\Big\{(t,s)\in[0,T]^2\bigm|0\le t\le s\le T\Big\}.$$

Recall from Section 1 that

$$\mathcal{U}[t,T]=\Big\{u:[t,T]\times\Omega\to U\ \big|\ u(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable}\Big\}.$$

Further, for $q\ge 1$, let

$$\mathcal{U}^q[t,T]=\Big\{u:[t,T]\times\Omega\to U\ \big|\ u(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable, }\mathbb{E}\int_t^T|u(s)|^q ds<\infty\Big\}.$$

Note that in the case U is bounded, for different $q\ge 1$, all the $\mathcal{U}^q[t,T]$ are the same as $\mathcal{U}[t,T]$. We introduce the following standing assumptions.

(H1) The maps $b:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^n$, $\sigma:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^{n\times d}$ are continuous and there exist constants $L>0$ and $k\ge 0$ such that

$$\left\{\begin{array}{ll}|b(t,x,u)-b(t,y,u)|\le L\big(1+(|x|\vee|y|)^k+|u|\big)|x-y|, & \\ \langle\, x-y,b(t,x,u)-b(t,y,u)\,\rangle\le L|x-y|^2, & \forall(t,u)\in[0,T]\times U,\ \ x,y\in\mathbb{R}^n,\\ |\sigma(t,x,u)-\sigma(t,y,u)|\le L|x-y|, & \end{array}\right. \tag{3.5}$$

where $|x|\vee|y|=\max\{|x|,|y|\}$, and

$$|b(t,0,u)|+|\sigma(t,0,u)|\le L(1+|u|),\qquad\qquad \forall(t,u)\in[0,T]\times U. \tag{3.6}$$

(H2) Maps $g:D[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}$ and $h:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ are continuous, and there exist constants $L>0$ and $q\ge 0$ such that

$$\left\{\begin{array}{l}0\le g(\tau,t,x,u)\le L\big(1+|x|^q+|u|^q\big),\\ 0\le h(\tau,x)\le L(1+|x|^q),\end{array}\right.\qquad \forall(\tau,t,x,u)\in D[0,T]\times\mathbb{R}^n\times U. \tag{3.7}$$

Let us make a couple of remarks on (H1). First of all, if $x\mapsto b(t,x,u)$ is uniformly Lipschitz, then the first two conditions of (3.5) hold. On the other hand, we point out that the first condition in (3.5) merely implies that $x\mapsto b(t,x,u)$ is locally Lipschitz, and the second condition in (3.5) alone does not imply the global Lipschitz condition for the map $x\mapsto b(t,x,u)$. A simple example that the first and the second conditions in (3.5) are satisfied but $x\mapsto b(t,x,u)$ is not uniform Lipschitz is the following:

$$b(t,x,u)=-|x|^2x-|u|x,\qquad (t,x,u)\in[0,T]\times\mathbb{R}^n\times U.$$

It is clear that the above map is not uniformly Lipschitz in x, the first condition in (3.5) holds with $k=2$, and we can check that

$$\langle\, x-y,b(t,x,u)-b(t,y,u)\,\rangle\le 0,\qquad \forall(t,u)\in[0,T]\times U,\ \ x,y\in\mathbb{R}^n.$$

Note that under (3.5), one has

$$\begin{aligned}\langle\, x,b(t,x,u)\,\rangle&=\langle\, x,b(t,x,u)-b(t,0,u)\,\rangle+\langle\, x,b(t,0,u)\,\rangle\\ &\le L|x|^2+L(1+|u|)|x|\le 2L\big(1+|x|^2+|u|^2\big),\end{aligned}$$

and

$$|\sigma(t,x,u)|^2\le L^2\big(1+|x|+|u|\big)^2\le 3L^2\big(1+|x|^2+|u|^2\big).$$

Now, for (H2), we note that the nonnegativity of $g(\cdot)$ and $h(\cdot)$ can be replaced by the condition that both $g(\cdot)$ and $h(\cdot)$ are bounded from below. The following result is concerning the well-posedness of the state equation.

Proposition 3.1. *Let* (H1) *hold. Then for any* $(t,x)\in[0,T]\times\mathbb{R}^n$ *and* $u(\cdot)\in\mathcal{U}^q[t,T]$, *with* $q\geq 2$, *state equation* (3.1) *admits a unique solution* $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))$, *and the following estimate hold:*

$$\mathbb{E}|X(s;t,x,u(\cdot))|^q\leq K\big(1+|x|^q+\int_t^s|u(r)|^qdr\big),\qquad s\in[t,T].\tag{3.8}$$

Moreover, if $\widehat{x}\in\mathbb{R}^n$ *is another point, the following holds:*

$$\mathbb{E}|X(s;t,x,u(\cdot)))-X(s;t,\widehat{x},u(\cdot))|^2\leq K|x-\widehat{x}|^2,\qquad \forall t\in[0,T],\ x,\widehat{x}\in\mathbb{R}^n,\ u(\cdot)\in\mathcal{U}^2[t,T].\tag{3.9}$$

Proof. Let $(t,x)\in[0,T)\times\mathbb{R}^n$ be given and $u(\cdot)\in\mathcal{U}^q[t,T]$. By the local Lipschitz condition, we see that the solution $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))$ exists locally, say on $[t,\tau)$ for some stopping time τ. Next, applying Itô's formula to $|X(\cdot)|^q$ on $[t,\tau)$, we have the following:

$$\begin{aligned}&\mathbb{E}|X(s)|^q=|x|^q+\mathbb{E}\int_t^s\Big(q|X(r)|^{q-2}\,\langle\,X(r),b(r,X(r),u(r))\,\rangle\\&+\frac{q(q-2)}{2}|X(r)|^{q-4}\,\langle\,X(r),\sigma(r,X(r),u(r))\,\rangle^{\,2}+\frac{q}{2}|X(r)|^{q-2}\mathrm{tr}\,\big[\sigma(s,X(s),u(s))\sigma(s,X(s),u(s))^T\big]\Big)dr\\&\leq|x|^q+K\mathbb{E}\int_t^s|X(r)|^{q-2}\big(1+|X(r)|^2+|u(r)|^2\big)dr\leq|x|^q+K\mathbb{E}\int_t^s\big(1+|X(r)|^q+|u(r)|^q\big)dr.\end{aligned}$$

By Gronwall's inequality, we obtain

$$\mathbb{E}|X(s)|^q\leq K\big(1+|x|^q+\mathbb{E}\int_t^s|u(r)|^qdr\big),\qquad s\in[t,\tau).$$

This implies that $X(\cdot)$ must be globally exists on $[t,T]$, and estimate (3.8) holds.

Next, for any $t\in[0,T)$, $x,\widehat{x}\in\mathbb{R}^n$, and $u(\cdot)\in\mathcal{U}^q[t,T]$, let $X(\cdot)=X(\cdot\,;t,x,u(\cdot))$ and $\widehat{X}(\cdot)=X(\cdot\,;t,\widehat{x},u(\cdot))$. Then applying Itô's formula to $|X(\cdot)-\widehat{X}(\cdot)|^2$, we get

$$\begin{aligned}\mathbb{E}|X(s)-\widehat{X}(s)|^2&=|x-\widehat{x}|^2+\mathbb{E}\int_t^s\Big(\langle\,X(s)-\widehat{X}(s),b(s,X(s),u(s))-b(s,\widehat{X}(s),u(s))\,\rangle\\&\qquad+|\sigma(s,X(s),u(s))-\sigma(s,\widehat{X}(s),u(s))|^2\Big)ds\\&\leq|x-\widehat{x}|^2+K\mathbb{E}\int_t^s|X(s)-\widehat{X}(s)|^2ds.\end{aligned}$$

Hence, applying Gronwall's inequality, we obtain (3.9). □

From the above result, we see that for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}^{q\vee2}[t,T]$, the following holds:

$$\begin{aligned}\mathbb{E}|g(\tau,s,X(s),u(s))|&\leq L\big(1+\mathbb{E}|X(s)|^{q\vee2}+\mathbb{E}|u(s)|^{q\vee2}\big)\\&\leq K\big(1+|x|^{q\vee2}+\mathbb{E}|u(s)|^{q\vee2}+\int_t^s\mathbb{E}|u(r)|^{q\vee2}dr\big),\end{aligned}$$

and

$$\mathbb{E}|h(\tau,X(T))|\leq L\big(1+\mathbb{E}|X(T)|^{q\vee2}\big)\leq K\big(1+|x|^{q\vee2}+\mathbb{E}\int_t^T|u(s)|^{q\vee2}ds\big).$$

Hence, $J(t,x;u(\cdot))$ is finite for any $u(\cdot)\in\mathcal{U}^{q\vee2}[t,T]$. For simplicity, hereafter, we adopt the following convention:

$$J(t,x;u(\cdot))\stackrel{\Delta}{=}+\infty,\qquad\text{if } J(t,x,u(\cdot)) \text{ is not finite or not defined.}\tag{3.10}$$

We now formally state our optimal control problem.

Problem (N). For any given initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$, find a $\bar u(\cdot)\in\mathcal{U}[t,T]$ such that

$$J(t,x;\bar u(\cdot))=\inf_{u(\cdot)\in\mathcal{U}[t,T]}J(t,x;u(\cdot)). \tag{3.11}$$

From the examples presented in the previous section, we know that the above Problem (N) is time-inconsistent, in general. Our goal is to find time-consistent equilibrium controls and characterize the equilibrium value function, which will be made precise below.

We denote

$$a(t,x,u)=\frac{1}{2}\sigma(t,x,u)\sigma(t,x,u)^T,\qquad\forall(t,x,u)\in[0,T]\times\mathbb{R}^n\times U.$$

Define

$$\begin{aligned}\mathbb{H}(\tau,t,x,u,p,P)=\langle\, b(t,x,u),p\,\rangle+\mathrm{tr}\,\big[a(t,x,u)P\big]+g(\tau,t,x,u),&\\ \forall(\tau,t,x,u,p,P)\in D[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathcal{S}^n,&\end{aligned} \tag{3.12}$$

and let

$$H(\tau,t,x,p,P)=\inf_{u\in U}\mathbb{H}(\tau,t,x,u,p,P),\qquad\forall(\tau,t,x,p,P)\in D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n. \tag{3.13}$$

Note that $H(\tau,t,x,p,P)$ is not necessarily finite on the whole space $D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n$ when U is unbounded. Therefore, we denote

$$\mathcal{D}(H)=\Big\{(\tau,t,x,p,P)\in D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n\;\big|\;H(\tau,t,x,p,P)>-\infty\Big\}, \tag{3.14}$$

call it the *domain* of H. Next, let

$$\begin{aligned}\arg\min\mathbb{H}(\tau,t,x,\cdot\,,p,P)=\Big\{\bar u\in U\;\big|\;\mathbb{H}(\tau,t,x,\bar u,p,P)=\min_{u\in U}\mathbb{H}(\tau,t,x,u,p,P)\Big\},&\\ \forall(\tau,t,x,p,P)\in\mathcal{D}(H).&\end{aligned} \tag{3.15}$$

This is a multi-valued map. Suppose we can define a map $\psi:\mathcal{D}(\psi)\subseteq\mathcal{D}(H)\to U$ such that

$$\begin{aligned}H(\tau,t,x,p,P)\equiv\mathbb{H}(\tau,t,x,\psi(\tau,t,x,p,P),p,P)=\inf_{u\in U}\mathbb{H}(\tau,t,x,u,p,P)>-\infty,&\\ \forall(\tau,t,x,p,P)\in\mathcal{D}(\psi).&\end{aligned} \tag{3.16}$$

The set $\mathcal{D}(\psi)$ is called the *domain* of ψ, which consists of all points $(\tau,t,x,p,P)\in\mathcal{D}(H)$ such that the infimum in (3.16) is achieved at $\psi(\tau,t,x,p,P)$. It is clear that $\psi(\cdot)$ is actually a selection of $\arg\min\mathbb{H}(\cdot)$, i.e.,

$$\psi(\tau,t,x,p,P)\in\arg\min\mathbb{H}(\tau,t,x,\cdot\,,p,P),\qquad\forall(\tau,t,x,p,P)\in\mathcal{D}(\psi). \tag{3.17}$$

The map $\psi(\cdot)$ will play an important role later. Therefore, let us say a little bit more on it. Note that when U is bounded (since it is assumed to be closed, it is compact in this case), one has

$$\mathcal{D}(\psi)=\mathcal{D}(H)=D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n.$$

However, when U is unbounded, say, $U=\mathbb{R}^m$, one might have

$$\mathcal{D}(\psi)\neq\mathcal{D}(H)\neq D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n. \tag{3.18}$$

To say something about the case of (3.18), let us present the following simple lemma.

Lemma 3.2. *Let $U\subseteq\mathbb{R}^m$ be closed and $f:U\to\mathbb{R}$ be a lower semi-continuous map such that*

$$\inf_{u\in U}f(u)\equiv\bar f>-\infty.$$

For any $\varepsilon > 0$, *let*

$$f^\varepsilon(u) = f(u) + \varepsilon|u|^2, \qquad u \in U.$$

Then there exists a $u^\varepsilon \in U$ *such that*

$$f^\varepsilon(u^\varepsilon) = \inf_{u\in U} f^\varepsilon(u) \equiv \bar f^\varepsilon.$$

Moreover,

$$\lim_{\varepsilon\downarrow 0} \bar f^\varepsilon = \bar f, \qquad \lim_{\varepsilon\downarrow 0} \varepsilon|u^\varepsilon|^2 = 0.$$

Proof. First of all, fix a $u_0 \in U$. For any minimizing sequence $u_k \in U$ of $f^\varepsilon(\cdot)$, we may assume that

$$f^\varepsilon(u_0) \geq f^\varepsilon(u_k) = f(u_k) + \varepsilon|u_k|^2 \geq \bar f + \varepsilon|u_k|^2, \qquad \forall k \geq 1.$$

Thus, u_k is bounded. Consequently, by the closeness of U, we may assume that $u_k \to u^\varepsilon \in U$ exists, which attains the infimum of $f^\varepsilon(\cdot)$. Next, it is clear that $f^\varepsilon(\cdot)$ decreases as ε decreases, and

$$\bar f \leq \lim_{\varepsilon\to 0} \bar f^\varepsilon \equiv \bar f^0.$$

Now, for any $\delta > 0$, there exists a $u_\delta \in U$ such that

$$f(u_\delta) < \bar f + \delta.$$

On the other hand,

$$\bar f^\varepsilon = f^\varepsilon(u^\varepsilon) \leq f^\varepsilon(u_\delta) = f(u_\delta) + \varepsilon|u_\delta|^2.$$

Hence, letting $\varepsilon \to 0$, we get

$$\bar f^0 \leq \bar f + \delta.$$

Since $\delta > 0$ is arbitrary, we must have $\bar f = \bar f^0$. Finally, by

$$\bar f \leq f(u^\varepsilon) \leq f(u^\varepsilon) + \varepsilon|u^\varepsilon|^2 = f^\varepsilon(u^\varepsilon) = \bar f^\varepsilon \to \bar f,$$

we see that

$$\lim_{\varepsilon\to 0} \varepsilon|u^\varepsilon|^2 = 0,$$

proving our conclusion. $\square$

Now, for the case $\mathcal{D}(\psi) \neq \mathcal{D}(H)$, we may introduce

$$\begin{aligned} &\mathbb{H}^\varepsilon(\tau,t,x,u,p,P) = \mathbb{H}(\tau,t,x,u,p,P) + \varepsilon|u|^2,\\ &\qquad\qquad\qquad\qquad (\tau,t,x,u,p,P) \in D[0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathcal{S}^n. \end{aligned} \tag{3.19}$$

Then by Lemma 3.2, one can find a $\psi^\varepsilon : \mathcal{D}(H) \to U$ such that

$$\begin{aligned} \mathbb{H}^\varepsilon(\tau,t,x,\psi^\varepsilon(\tau,t,x,p,P),p,P) &= \inf_{u\in U} \mathbb{H}^\varepsilon(\tau,t,x,u,p,P)\\ &\equiv H^\varepsilon(\tau,t,x,p,P), \qquad\qquad (\tau,t,x,p,P) \in \mathcal{D}(H). \end{aligned} \tag{3.20}$$

Moreover, $\varepsilon \mapsto H^\varepsilon(\tau,t,x,p,P)$ is decreasing as $\varepsilon\downarrow 0$, and

$$\begin{cases} \lim\limits_{\varepsilon\to 0} H^\varepsilon(\tau,t,x,p,P) = H(\tau,t,x,p,P), \\ \lim\limits_{\varepsilon\to 0} \varepsilon|\psi^\varepsilon(\tau,t,x,p,P)|^2 = 0, \end{cases} \qquad (\tau,t,x,p,P) \in \mathcal{D}(H).$$

In general, we do not expect the convergence of $\psi^\varepsilon(\tau,t,x,p,P)$ as $\varepsilon \to 0$.

We point out that in general the map $\psi : \mathcal{D}(\psi) \subseteq D[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to U$ is not necessarily continuous. Here is a simple example.

Example 3.3. Let $n = m = d = 1$, $U = [0,1]$, and

$$b(t,x,u) = u, \quad \sigma(t,x,u) = \sigma(t,x), \quad g(\tau,t,x,u) = R(\tau,t)u,$$

with $\sigma(\cdot\,,\cdot)$ and $R(\cdot\,,\cdot)$ being continuous, and $R(\tau,t) > 0$ for all $(\tau,t) \in D[0,T]$. Then

$$\mathbb{H}(\tau,t,x,u,p,P) = pu + \frac{1}{2}\sigma(t,x)^2 P + R(\tau,t)u = \frac{1}{2}\sigma(t,x)^2 P + \big[p + R(\tau,t)\big]u.$$

Consequently,

$$H(\tau,t,x,p,P) = \frac{1}{2}\sigma(t,x)^2 P + \min\{p + R(\tau,t), 0\}, \qquad (\tau,t,x,p,P) \in D[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n,$$

which is continuous, and

$$\psi(\tau,t,x,p,P) = I_{(p+R(\tau,t)<0)}, \qquad \forall (\tau,t,x,p,P) \in D[0,T] \times \mathbb{R}^3,$$

which has discontinuity along $p + R(\tau,t) = 0$.

The following example shows that sometimes, ψ can also be continuous.

Example 3.4. Let $n = m = d = 1$, $U = [-1,1]$, and

$$b(t,x,u) = u, \quad \sigma(t,x,u) = \sigma(t,x), \quad g(\tau,t,x,u) = \frac{1}{2}R(\tau,t)u^2,$$

with $\sigma(\cdot\,,\cdot)$ and $R(\cdot\,,\cdot)$ being continuous, and $R(\tau,t) > 0$ for all $(\tau,t) \in D[0,T]$. Then

$$\begin{aligned}\mathbb{H}(\tau,t,x,u,p,P) &= pu + \frac{1}{2}\sigma(t,x)^2 P + \frac{1}{2}R(\tau,t)u^2 \\ &= \frac{1}{2}\sigma(t,x)^2 P + \frac{1}{2}R(\tau,t)\Big(u + \frac{p}{R(\tau,t)}\Big)^2 - \frac{p^2}{2R(\tau,t)}.\end{aligned}$$

Consequently,

$$H(\tau,t,x,p,P) = \frac{1}{2}\sigma(t,x)^2 P + \frac{1}{2}R(\tau,t)\Big(\frac{|p| \wedge [R(\tau,t)]}{R(\tau,t,x)} - \frac{|p|}{R(\tau,t)}\Big)^2 - \frac{p^2}{2R(\tau,t)},$$

and

$$\psi(\tau,t,x,p,P) = -\big[\operatorname{sgn} p\big]\Big(\frac{|p|}{R(\tau,t)} \wedge 1\Big), \qquad \forall (\tau,t,x,p,P) \in D[0,T] \times \mathbb{R}^3.$$

Clearly, both H and ψ are continuous. Also, we see that even if all the coefficients are very smooth, we cannot guarantee that H and ψ are as smooth as the coefficients, in general.

Here is another example which shows that H and ψ could be as smooth as the coefficients.

Example 3.5. Let $n = m = d = 1$, $U = (-1,1)$, and

$$b(t,x,u) = u, \quad \sigma(t,x,u) = \sigma(t,x), \quad g(\tau,t,x,u) = -R(\tau,t,x)\ln(1-u^2),$$

with $R(\tau,t,x)$ being positive-valued and bounded. Then

$$\mathbb{H}(\tau,t,x,u,p,P) = pu + \frac{1}{2}\sigma(t,x)^2 P - R(\tau,t,x)\ln(1-u^2).$$

A direct computation shows that

$$
\begin{aligned}
H(\tau,t,x,p,P) &= p\psi(\tau,t,x,p,P)+\frac{1}{2}\sigma(t,x)^2P-R(\tau,t,x)\ln\big[1-\psi(\tau,t,x,p,P)^2\big]\\
&=\frac{1}{2}\sigma(t,x)^2P-\frac{p^2}{R(\tau,t,x)+\sqrt{R(\tau,t,x)^2+p^2}}+R(\tau,t,x)\ln\frac{R(\tau,t,x)+\sqrt{R(\tau,t,x)^2+p^2}}{2R(\tau,t,x)},
\end{aligned}
$$

with

$$
\psi(\tau,t,x,p,P)=\frac{-p}{R(\tau,t,x)+\sqrt{R(\tau,t,x)^2+p^2}}.
$$

Therefore, both H and ψ are as smooth as the coefficients.

From the above discussion, we see that the situation concerning the map $\psi(\cdot)$ is very complicated. For the simplicity of presentation below, we adopt the following assumption.

(H3) The map $\psi: D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n\to U$ is well-defined and has needed regularity.

We will address more general situations concerning $\psi(\cdot)$ in our future publications.

Now, let us recall a standard verification theorem for Problem (C) stated in Section 1, which will be used below. For our later purposes, it suffices to consider Problem (C) with the discount rate $\delta=0$. We will denote

$$
C^{1,2}([0,T]\times\mathbb{R}^n)=\Big\{v(\cdot\,,\cdot)\in C([0,T]\times\mathbb{R}^n)\bigm| v_t(\cdot\,,\cdot),v_x(\cdot\,,\cdot),v_{xx}(\cdot\,,\cdot)\in C([0,T]\times\mathbb{R}^n)\Big\}.
$$

The proof of the following result can be found in [8].

Proposition 3.6. *Suppose $V^0(\cdot\,,\cdot)\in C^{1,2}([0,T]\times\mathbb{R}^n)$ is a classical solution to the following Hamilton-Jacobi-Bellman equation:*

$$
\begin{cases}
V^0_t(t,x)+\inf\limits_{u\in U}\mathbb{H}^0\big(t,x,u,V^0_x(t,x),V^0_{xx}(t,x)\big)=0, \qquad (t,x)\in[0,T]\times\mathbb{R}^n,\\
V^0(T,x)=h^0(x), \qquad x\in\mathbb{R}^n,
\end{cases}
\tag{3.21}
$$

where

$$
\mathbb{H}^0(t,x,u,p,P)=\langle\, b(t,x,u),p\,\rangle+\mathrm{tr}\,\big[a(t,x,u)P\big]+g^0(t,x,u). \tag{3.22}
$$

Then

$$
V^0(t,x)\le J^0(t,x;u(\cdot)), \qquad \forall(t,x)\in[0,T]\times\mathbb{R}^n,\ u(\cdot)\in\mathcal{U}[t,T]. \tag{3.23}
$$

If $(t,x)\in[0,T)\times\mathbb{R}^n$ is given and $(\bar X(\cdot),\bar u(\cdot))$ is a state-control pair with the initial pair (t,x) such that

$$
\bar u(s)\in\arg\min\mathbb{H}^0\big(s,\bar X(s),\cdot\,,V^0_x(s,\bar X(s)),V^0_{xx}(s,\bar X(s))\big), \quad s\in[t,T], \tag{3.24}
$$

then

$$
V^0(t,x)=J^0(t,x;\bar u(\cdot)) \tag{3.25}
$$

and $(\bar X(\cdot),\bar u(\cdot))$ is an optimal pair of Problem (C) *for the initial pair (t,x).*

We make some remarks on the above verification theorem. First of all, to guarantee (3.21) to have a classical solution $V^0(\cdot\,,\cdot)$, one may pose different conditions. A typical one is the uniform ellipticity condition:

$$
a(t,x,u)\equiv\frac{1}{2}\sigma(t,x,u)\sigma(t,x,u)^T\ge\delta I_n, \qquad \forall(t,x,u)\in[0,T]\times\mathbb{R}^n\times U, \tag{3.26}
$$

for some $\delta>0$. This condition implies that $n\le d$ and $\sigma(t,x,u)$ stays full rank for all (t,x,u). Thus, it does not include the case that $(x,u)\mapsto\sigma(t,x,u)$ is linear, which is the case for LQ problems. On the other hand, for a standard LQ problem with deterministic coefficients, when the Riccati equation admits a solution $P(\cdot)$, the function $V^0(t,x)=\langle\, P(t)x,x\,\rangle$ is a classical solution to the corresponding HJB equation for which the

uniform ellipticity condition fails. The similar situation happens for the classical Merton's portfolio problem. This observation shows that there are quite different conditions under which the corresponding HJB equation admits a classical solution. In the following sections, from time to time, we will simply say that the relevant HJB equation has a classical solution without getting into detailed sufficient conditions for that. Likewise, suppose there exists a map $\psi^0 : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to U$ such that

$$\mathbb{H}^0\big(s,x,\psi^0(s,x,p,P),p,P\big) = \inf_{u\in U} \mathbb{H}^0(s,x,u,p,P), \qquad \forall (s,x,p,P) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n.$$

Then the pair $(\bar{X}(\cdot), \bar{u}(\cdot))$ appears in Proposition 3.6 is just a state-control pair satisfying the following

$$\begin{cases} d\bar{X}(s) = b\big(s,\bar{X}(s),\psi^0(s,\bar{X}(s),V_x^0(s,\bar{X}(s)),V_{xx}^0(s,\bar{X}(s))\big)ds \\ \qquad\qquad +\sigma\big(s,\bar{X}(s),\psi^0(s,\bar{X}(s),V_x^0(s,\bar{X}(s)),V_{xx}^0(s,\bar{X}(s))\big)sW(s), \qquad s\in[t,T], \\ \bar{X}(t) = x, \end{cases} \tag{3.27}$$

and we do not have to have the Lipschitz continuity of the map

$$x \mapsto \Big(b\big(s,x,\psi^0(s,x,V_x^0(s,x),V_{xx}^0(s,x))\big), \sigma\big(s,x,\psi^0(s,x,V_x^0(s,x),V_{xx}^0(s,x))\big)\Big).$$

In the following section, from time to time, we will just say some process is a solution to the relevant closed-loop system without mentioning if the drift and diffusion are Lipschitz continuous, etc.

4 Time-Consistent Equilibria via Multi-Person Differential Games

In this section, we are going to search time-consistent solution to Problem (N). Inspired by [27, 28], we will take an approach of multi-person differential games.

To begin with, let us first introduce some necessary notions. Let $\mathcal{P}[0,T]$ be the set of all partitions $\Pi = \{t_k \bigm| 0 \le k \le N\}$ of $[0,T]$ with $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$, and with the mesh size $\|\Pi\|$ defined by the following:

$$\|\Pi\| = \max_{1\le k\le N}(t_k - t_{k-1}).$$

We introduce the following important definition.

Definition 4.1. A continuous map $\Psi : [0,T] \times \mathbb{R}^n \to U$ is called a *time-consistent equilibrium strategy* of Problem (N) if for any $(t,x) \in [0,T) \times \mathbb{R}^n$,

$$\begin{cases} d\bar{X}(s) = b\big(s,\bar{X}(s),\Psi(s,\bar{X}(s))\big)ds + \sigma\big(s,\bar{X}(s),\Psi(s,\bar{X}(s))\big)dW(s), \qquad s\in[t,T], \\ \bar{X}(t) = x, \end{cases} \tag{4.1}$$

admits a unique solution $\bar{X}(\cdot) \equiv \bar{X}(\cdot\,; t, x, \Psi(\cdot))$ such that for any $t \in [0,T)$, and $\varepsilon > 0$ with $t + \varepsilon \le T$,

$$\begin{aligned} &J\Big(t,\bar{X}\big(t;0,x,\Psi(\cdot)\big);\Psi(\cdot)\big|_{[t,T]}\Big) \\ &\le J\Big(t,\bar{X}\big(t;0,x,\Psi(\cdot)\big);u(\cdot)\oplus\Psi(\cdot)\big|_{[t+\varepsilon,T]}\Big) + R(\varepsilon), \qquad \forall u(\cdot)\in\mathcal{U}[t,t+\varepsilon], \end{aligned} \tag{4.2}$$

where $R(\varepsilon)$ represents a generic remainder term satisfying $R(\varepsilon) \to 0$ as $\varepsilon \to 0$, and

$$\big[u(\cdot)\oplus\Psi(\cdot)\big|_{[t+\varepsilon,T]}\big](s) = \begin{cases} u(s), & t\in[t,t+\varepsilon), \\ \Psi\big(s,X^\varepsilon(s)\big), & s\in[t+\varepsilon,T], \end{cases} \tag{4.3}$$

with

$$\begin{cases} dX^\varepsilon(s) = b\big(s,X^\varepsilon(s),u(s)\big)ds + \sigma\big(s,X^\varepsilon(s),u(s)\big)dW(s), \qquad s\in[t,t+\varepsilon), \\ dX^\varepsilon(s) = b\big(s,X^\varepsilon(s),\Psi(s,X^\varepsilon(s))\big)ds + \sigma\big(s,X^\varepsilon(s),\Psi(s,X^\varepsilon(s))\big)dW(s), \qquad s\in[t+\varepsilon,T], \\ X^\varepsilon(t) = \bar{X}\big(t;0,x,\Psi(\cdot)\big). \end{cases} \tag{4.4}$$

In this case, we call $\bar{X}(\cdot) \equiv \bar{X}(\cdot\,; 0, x, \Psi(\cdot))$ a *time-consistent equilibrium state process*, call the corresponding $\bar{u}(\cdot) \equiv \Psi\big(\cdot\,; \bar{X}(\cdot)\big)$ a *time-consistent equilibrium control* for the initial state x, and call $(\bar{X}(\cdot), \bar{u}(\cdot))$ a *time-consistent equilibrium pair*. Further, function $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is called the *equilibrium value function* of Problem (N) if

$$V\big(t, \bar{X}(t; 0, x, \Psi(\cdot))\big) = J\big(t, \bar{X}(t; 0, x, \Psi(\cdot)); \Psi(\cdot)\big|_{[t,T]}\big), \qquad t \in [0, T]. \tag{4.5}$$

Note that our definition of time-consistent equilibrium strategy is of closed-loop type. Thus, it is comparable with that given in [5, 2, 3], and it is different from that in [13].

In what follows, the following definition which is equivalent to the above is more convenient to use.

Definition 4.1′. A continuous map $\Psi : [0, T] \times \mathbb{R}^n \to U$ is called a *time-consistent equilibrium strategy* of Problem (N) if for any $(t, x) \in [0, T) \times \mathbb{R}^n$,

$$\begin{cases} d\bar{X}(s) = b\big(s, \bar{X}(s), \Psi(s, \bar{X}(s))\big)ds + \sigma\big(s, \bar{X}(s), \Psi(s, \bar{X}(s))\big)dW(s), \qquad s \in [t, T], \\ \bar{X}(t) = x, \end{cases} \tag{4.6}$$

admits a unique solution $\bar{X}(\cdot) \equiv \bar{X}(\cdot\,; t, x, \Psi(\cdot))$ such that for any $\Pi \equiv \{t_k \bigm| 1 \le k \le N\} \in \mathcal{P}[0, T]$,

$$\begin{aligned} & J\Big(t_{k-1}, \bar{X}\big(t_{k-1}; 0, x, \Psi(\cdot)\big); \Psi(\cdot)\big|_{[t_{k-1},T]}\Big) \\ & \le J\Big(t_{k-1}, \bar{X}\big(t_{k-1}; 0, x, \Psi(\cdot)\big); u^k(\cdot) \oplus \Psi(\cdot)\big|_{[t_k,T]}\Big) + R\big(\|\Pi\|\big), \\ & \hspace{6cm} \forall u^k(\cdot) \in \mathcal{U}[t_{k-1}, t_k], \end{aligned} \tag{4.7}$$

where $R(r)$ represents a generic remainder term satisfying $R(r) \to 0$ as $r \to 0$, and

$$\big[u^k(\cdot) \oplus \Psi(\cdot)\big|_{[t_k,T]}\big](t) = \begin{cases} u^k(t), & t \in [t_{k-1}, t_k), \\ \Psi\big(t, X^k(t)\big), & t \in [t_k, T], \end{cases} \tag{4.8}$$

with

$$\begin{cases} dX^k(s) = b\big(s, X^k(s), u^k(s)\big)ds + \sigma\big(s, X^k(s), u^k(s)\big)dW(s), \qquad s \in [t_{k-1}, t_k), \\ dX^k(s) = b\big(s, X^k(s), \Psi(s, X^k(s))\big)ds + \sigma\big(s, X^k(s), \Psi(s, X^k(s))\big)dW(s), \qquad s \in [t_k, T], \\ X^k(t_{k-1}) = \bar{X}\big(t_{k-1}; 0, x, \Psi(\cdot)\big). \end{cases} \tag{4.9}$$

4.1 Multi-Person Differential Games.

We now consider an N-person differential game, called Problem (G^Π), as briefly described in Section 1. Throughout this section, we assume that (H1)–(H3) hold. Let us start with Player N who controls the system on $[t_{N-1}t_N)$. More precisely, for each $(t, x) \in [t_{N-1}, t_N] \times \mathbb{R}^n$, consider the following controlled SDE:

$$\begin{cases} dX^N(s) = b\big(s, X^N(s), u^N(s)\big)ds + \sigma\big(s, X^N(s), u^N(s)\big)dW(s), \qquad s \in [t, t_N], \\ X^N(t) = x, \end{cases} \tag{4.10}$$

with cost functional

$$J^N(t, x; u^N(\cdot)) = \mathbb{E}_t\Big[\int_t^{t_N} g\big(t_{N-1}, s, X^N(s), u^N(s)\big)ds + h\big(t_{N-1}, X^N(t_N)\big)\Big]. \tag{4.11}$$

Note that

$$J^N(t_{N-1}, x; u^N(\cdot)) = J(t_{N-1}, x; u^N(\cdot)), \qquad (x, u^N(\cdot)) \in \mathbb{R}^n \times \mathcal{U}[t_{N-1}, t_N]. \tag{4.12}$$

We pose the following problem for Player N:

Problem (C$_N$). For any $(t,x)\in[t_{N-1},t_N)\times\mathbb{R}^n$, find a $\bar u^N(\cdot)\equiv\bar u^N(\cdot\,;t,x)\in\mathcal{U}[t,t_N]$ such that

$$J^N(t,x;\bar u^N(\cdot))=\inf_{u^N(\cdot)\in\mathcal{U}[t,t_N]}J^N(t,x;u^N(\cdot))\equiv V^\Pi(t,x). \tag{4.13}$$

The above defines the *value function* $V^\Pi(\cdot\,,\cdot)$ on $[t_{N-1},t_N]\times\mathbb{R}^n$, and in the case $\bar u^N(\cdot)$ exists, by (4.12), we have

$$J(t_{N-1},x;\bar u^N(\cdot))=V^\Pi(t_{N-1},x),\qquad\forall x\in\mathbb{R}^n. \tag{4.14}$$

Under proper conditions, $V^\Pi(\cdot\,,\cdot)$ is the classical solution to the following HJB equation:

$$\begin{cases}V^\Pi_t(t,x)+\inf\limits_{u\in U}\mathbb{H}\big(t_{N-1},t,x,u,V^\Pi_x(t,x),V^\Pi_{xx}(t,x)\big)=0,\qquad(t,x)\in[t_{N-1},t_N]\times\mathbb{R}^n,\\ V^\Pi(t_N,x)=h(t_{N-1},x),\qquad x\in\mathbb{R}^n.\end{cases} \tag{4.15}$$

By the definition of $\psi:D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n\to U$ (see (3.16)–(3.17)), we may also write (4.15) as follows

$$\begin{cases}V^\Pi_t(t,x)+\mathbb{H}\big(t_{N-1},t,x,\psi(t_{N-1},t,x,V^\Pi_x(t,x),V^\Pi_{xx}(t,x)),V^\Pi_x(t,x),V^\Pi_{xx}(t,x)\big)=0,\\ \qquad\qquad\qquad\qquad(t,x)\in[t_{N-1},t_N]\times\mathbb{R}^n,\\ V^\Pi(t_N,x)=h(t_{N-1},x),\qquad x\in\mathbb{R}^n.\end{cases} \tag{4.16}$$

Clearly, $V^\Pi(\cdot\,,\cdot)$, well-defined on $[t_{N-1},t_N]\times\mathbb{R}^n$, depends on t_{N-1} and t_N. With such a solution $V^\Pi(\cdot\,,\cdot)$ of (4.15) (or (4.16)), let us assume that the following closed-loop system admits a unique solution $\bar X^N(\cdot)\equiv\bar X^N(\cdot\,;t_{N-1},x)$: (suppressing the dependence of $\bar X^N(\cdot)$ on t_N through $V^\Pi(\cdot\,,\cdot)$)

$$\begin{cases}d\bar X^N(s)=b\big(s,\bar X^N(s),\psi(t_{N-1},s,\bar X^N(s),V^\Pi_x(s,\bar X^N(s)),V^\Pi_{xx}(s,\bar X^N(s)))\big)ds\\ \qquad\qquad+\sigma\big(s,\bar X^N(s),\psi(t_{N-1},s,\bar X^N(s),V^\Pi_x(s,\bar X^N(s)),V^\Pi_{xx}(s,\bar X^N(s)))\big)dW(s),\\ \qquad\qquad\qquad\qquad s\in[t_{N-1},t_N],\\ \bar X^N(t_{N-1})=x.\end{cases} \tag{4.17}$$

Then under (H3) and Proposition 3.6, an optimal control $\bar u^N(\cdot)$ of Problem (C$_N$) for the initial pair (t_{N-1},x) admits the following feedback representation:

$$\begin{aligned}\bar u^N(s)\equiv\bar u^N(s;t_{N-1},x)&=\psi\big(t_{N-1},s,\bar X^N(s),V^\Pi_x(s,\bar X^N(s)),V^\Pi_{xx}(s,\bar X^N(s))\big)\\ &\equiv\psi\big(t_{N-1},s,\bar X^N(s;t_{N-1},x),V^\Pi_x(s,\bar X^N(s;t_{N-1},x)),V^\Pi_{xx}(s,\bar X^N(s;t_{N-1},x))\big)\\ &\qquad\qquad s\in[t_{N-1},t_N],\end{aligned} \tag{4.18}$$

and $\bar X^N(\cdot)\equiv\bar X^N(\cdot\,;t_{N-1},x)$ is the corresponding optimal state process.

Next, we consider an optimal control problem for Player $(N-1)$ on $[t_{N-2},t_{N-1})$. Naturally, for any initial pair $(t,x)\in[t_{N-2},t_{N-1}]\times\mathbb{R}^n$, the state equation should be

$$\begin{cases}dX^{N-1}(s)=b\big(s,X^{N-1}(s),u^{N-1}(s)\big)ds+\sigma\big(s,X^{N-1}(s),u^{N-1}(s)\big)dW(s),\quad s\in[t,t_{N-1}),\\ X^{N-1}(t)=x.\end{cases} \tag{4.19}$$

To determine the suitable cost functional, we note that Player $(N-1)$ can only control the system on $[t_{N-2},t_{N-1})$ and Player N will take over at t_{N-1} to control the system thereafter. Moreover, Player $(N-1)$ knows that Player N will play optimally based on the initial pair $(t_{N-1},X^{N-1}(t_{N-1}))$ of Player N, which is the *terminal pair* of Player $(N-1)$. Hence, the *sophisticated cost functional* of Player $(N-1)$ should be

$$\begin{aligned}J^{N-1}(t,x;u^{N-1}(\cdot))=\mathbb{E}_t\Big[&\int_t^{t_{N-1}}g(t_{N-2},s,X^{N-1}(s),u^{N-1}(s))ds\\ &+\int_{t_{N-1}}^{t_N}g\big(t_{N-2},s,\bar X^N(s;t_{N-1},X^{N-1}(t_{N-1})),\bar u^N(s;t_{N-1},X^{N-1}(t_{N-1}))\big)ds\\ &+h\big(t_{N-2},\bar X^N(t_N;t_{N-1},X^{N-1}(t_{N-1}))\big)\Big].\end{aligned} \tag{4.20}$$

Note that although Player $(N-1)$ knows that Player N will control the system on $[t_{N-1},t_N]$, he/she still "discounts" the future costs in his/her own way (see t_{N-2} appearing in the running cost on $[t_{N-1},t_N]$ and in the terminal cost at t_N). Now, if we denote

$$h^{N-1}(x)=\mathbb{E}_{t_{N-1}}\Big[\int_{t_{N-1}}^{t_N}g\big(t_{N-2},s,\bar X^N(s;t_{N-1},x),\bar u^N(s;t_{N-1},x)\big)ds+h\big(t_{N-2},\bar X^N(t_N;t_{N-1},x)\big)\Big],$$

then the cost functional (4.20) can be written as

$$J^{N-1}(t,x;u^{N-1}(\cdot))=\mathbb{E}_t\Big[\int_t^{t_{N-1}}g\big(t_{N-2},s,X^{N-1}(s),u^{N-1}(s)\big)ds+h^{N-1}\big(X^{N-1}(t_{N-1})\big)\Big].\tag{4.21}$$

We see that the optimal control problem associated with the state equation (4.19) and the cost functional (4.21) looks like a standard one. But, the map $x\mapsto h^{N-1}(x)$ seems to be a little too implicit, which is difficult for us to pass to the limits later on. We now would like to make it more explicit in some sense. Inspired by the idea of Four Step Scheme introduced in [16, 17] for FBSDEs with deterministic coefficients, we proceed as follows. For the optimal state process $\bar X^N(\cdot)\equiv\bar X^N(\cdot\,;t_{N-1},x)$ determined by (4.17) on $[t_{N-1},t_N]$, we introduce the following BSDE:

$$\begin{cases}dY^N(s)=-g\big(t_{N-2},s,\bar X^N(s),\psi(t_{N-1},s,\bar X^N(s),V_x^\Pi(s,\bar X^N(s)),V_{xx}^\Pi(s,\bar X^N(s)))\big)ds\\ \qquad\qquad\qquad\qquad +Z^N(s)dW(s),\qquad\qquad s\in[t_{N-1},t_N],\\ Y^N(t_N)=h(t_{N-2},\bar X^N(t_N)),\end{cases}\tag{4.22}$$

which is equivalent to the following:

$$\begin{cases}dY^N(s)=-g\big(t_{N-2},s,\bar X^N(s),\bar u^N(s)\big)ds+Z^N(s)dW(s),\qquad s\in[t_{N-1},t_N],\\ Y^N(t_N)=h(t_{N-2},\bar X^N(t_N)),\end{cases}\tag{4.23}$$

Note that t_{N-2} appears in the drift of BSDE and in the terminal condition. This BSDE admits a unique adapted solution $(Y^N(\cdot),Z^N(\cdot))\equiv\big(Y^N(\cdot\,;x),Z^N(\cdot\,;x)\big)$ ([17, 29]), uniquely depending on $x\in\mathbb{R}^n$. Further, one has

$$Y^N(t_{N-1})=\mathbb{E}_{t_{N-1}}\Big[\int_{t_{N-1}}^{t_N}g\big(t_{N-2},s,\bar X^N(s),\bar u^N(s)\big)ds+h(t_{N-2},\bar X^N(t_N))\Big]=h^{N-1}(x).$$

It is seen that (4.17) and (4.23) form an FBSDE. By [16] (see also [17, 29]), we have the following representation for $Y^N(\cdot)$

$$Y^N(s)=\Theta^N(s,\bar X^N(s)),\qquad s\in[t_{N-1},t_N],\tag{4.24}$$

as long as $\Theta^N(\cdot\,,\cdot)$ is a classical solution to the following PDE:

$$\begin{cases}\Theta_s^N(s,x)+\langle\,\Theta_x^N(s,x),b\big(s,x,\psi(t_{N-1},s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\,\rangle\\ \qquad+\mathrm{tr}\,\big[a\big(s,x,\psi(t_{N-1},s,x,V_x^\Pi(s,x),V_{xx}^N(s,x))\big)\Theta_{xx}^N(s,x)\big]\\ \qquad+g\big(t_{N-2},s,x,\psi(t_{N-1},s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)=0,\qquad(s,x)\in[t_{N-1},t_N]\times\mathbb{R}^n,\\ \Theta^N(t_N,x)=h(t_{N-2},x),\qquad x\in\mathbb{R}^n,\end{cases}\tag{4.25}$$

or equivalently,

$$\begin{cases}\Theta_s^N(s,x)+\mathbb{H}\big(t_{N-2},s,x,\psi\big(t_{N-1},s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x)\big),\Theta_x^N(s,x),\Theta_{xx}^N(s,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad(s,x)\in[t_{N-1},t_N]\times\mathbb{R}^n,\\ \Theta^N(t_N,x)=h(t_{N-2},x),\qquad x\in\mathbb{R}^n.\end{cases}\tag{4.26}$$

Note that $\Theta^N(\cdot\,,\cdot)$ depends on (t_{N-2},t_{N-1},t_N). We point out that in general,

$$\begin{aligned}&\Theta^N(t_{N-1},x)=Y^N(t_{N-1})=\mathbb{E}_{t_{N-1}}\Big[\int_{t_{N-1}}^{t_N}g\big(t_{N-2},s,\bar X^N(s),\bar u^N(s)\big)ds+h(t_{N-2},\bar X^N(T))\Big]=h^{N-1}(x)\\&\neq\mathbb{E}_{t_{N-1}}\Big[\int_{t_{N-1}}^{t_N}g\big(t_{N-1},s,\bar X^N(s),\bar u^N(s)\big)ds+h(t_{N-1},\bar X^N(T))\Big]=V^\Pi(t_{N-1},x).\end{aligned}\tag{4.27}$$

With the above representation $\Theta^N(\cdot\,,\cdot)$ of $Y^N(\cdot)$, we can rewrite the cost functional (4.21) as follows:

$$J^{N-1}(t,x;u^{N-1}(\cdot))=\mathbb{E}_t\Big[\int_t^{t_{N-1}}g(t_{N-2},s,X^{N-1}(s),u^{N-1}(s))ds+\Theta^N(t_{N-1},X^{N-1}(t_{N-1}))\Big].\tag{4.28}$$

We now pose the following problem:

Problem (C_{N-1}). For any $(t,x)\in[t_{N-2},t_{N-1})\times\mathbb{R}^n$, find a $\bar u^{N-1}(\cdot)\equiv\bar u^{N-1}(\cdot\,;t,x)\in\mathcal{U}[t_{N-2},t_{N-1}]$ such that

$$J^{N-1}(t,x;\bar u^{N-1}(\cdot))=\inf_{u^{N-1}(\cdot)\in\mathcal{U}[t,t_{N-1}]}J^{N-1}(t,x;u^{N-1}(\cdot))\equiv V^\Pi(t,x).\tag{4.29}$$

The above defines the value function $V^\Pi(\cdot\,,\cdot)$ on $[t_{N-2},t_{N-1})\times\mathbb{R}^n$. Under proper conditions, $V^\Pi(\cdot\,,\cdot)$ is the classical solution to the following HJB equation:

$$\begin{cases}V_t^\Pi(t,x)+\inf\limits_{u\in U}\mathbb{H}\big(t_{N-2},t,x,u,V_x^\Pi(t,x),V_{xx}^\Pi(t,x)\big)=0,\qquad(t,x)\in[t_{N-2},t_{N-1})\times\mathbb{R}^n,\\ V^\Pi(t_{N-1}-0,x)=\Theta^N(t_{N-1},x),\qquad x\in\mathbb{R}^n.\end{cases}\tag{4.30}$$

By the definition of the map $\psi(\cdot)$ again (see (3.16)–(3.17)), we may also write the above as

$$\begin{cases}V_t^\Pi(t,x)+\mathbb{H}\big(t_{N-2},t,x,\psi\big(t_{N-2},t,x,V_x^\Pi(t,x),V_{xx}^\Pi(t,x)\big),V_x^\Pi(t,x),V_{xx}^\Pi(t,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad(t,x)\in[t_{N-2},t_{N-1})\times\mathbb{R}^n,\\ V^\Pi(t_{N-1}-0,x)=\Theta^N(t_{N-1},x),\qquad x\in\mathbb{R}^n.\end{cases}\tag{4.31}$$

From (4.27), we see that in general,

$$V^\Pi(t_{N-1}-0,x)=\Theta^N(t_{N-1},x)\neq V^\Pi(t_{N-1},x).$$

Thus, $V^\Pi(\cdot\,,\cdot)$, which is now defined on $[t_{N-2},t_N]\times\mathbb{R}^n$, may have a jump at $\{t_{N-1}\}\times\mathbb{R}^n$. For any $x\in\mathbb{R}^n$, suppose the following admits a unique solution $\bar X^{N-1}(\cdot)$:

$$\begin{cases}d\bar X^{N-1}(s)=b\Big(s,\bar X^{N-1}(s),\psi\big(t_{N-2},s,\bar X^{N-1}(s),V_x^\Pi(s,\bar X^{N-1}(s)),V_{xx}^\Pi(s,\bar X^{N-1}(s))\big)\Big)ds\\ \qquad\qquad+\sigma\Big(s,\bar X^{N-1}(s),\psi\big(t_{N-2},s,\bar X^{N-1}(s),V_x^\Pi(s,\bar X^{N-1}(s)),V_{xx}^\Pi(s,\bar X^{N-1}(s))\big)\Big)dW(s),\\ \qquad\qquad\qquad\qquad\qquad\qquad s\in[t_{N-2},t_{N-1}),\\ \bar X^{N-1}(t_{N-2})=x.\end{cases}\tag{4.32}$$

Then we define

$$\begin{aligned}\bar u^{N-1}(s;t_{N-2},x)&=\psi\big(t_{N-2},s,\bar X^{N-1}(s),V_x^\Pi(s,\bar X^{N-1}(s)),V_{xx}^\Pi(s,\bar X^{N-1}(s))\big),\\&\equiv\psi\big(t_{N-2},s,\bar X^{N-1}(s;t_{N-2},x),V_x^\Pi(s,\bar X^{N-1}(s;t_{N-2},x)),V_{xx}^\Pi(s,\bar X^{N-1}(s;t_{N-2},x))\big),\\&\qquad\qquad\qquad\qquad s\in[t_{N-2},t_{N-1}),\end{aligned}$$

which, again by Proposition 3.6, is an optimal control of Problem (C_{N-1}) with the initial pair (t_{N-2},x). Now, for the optimal pair

$$\big(\bar X^{N-1}(\cdot),\bar u^{N-1}(\cdot)\big)=\big(\bar X^{N-1}(\cdot\,;t_{N-2},x),\bar u^{N-1}(\cdot\,;t_{N-2},x)\big)$$

of Problem (C_{N-1}) (on $[t_{N-2},t_{N-1}]$), we make a natural extension on $[t_{N-1},t_N]$ as follows:

$$\begin{cases} \bar X^{N-1}(s)=\bar X^N(s;t_{N-1},\bar X^{N-1}(t_{N-1})),\\ \bar u^{N-1}(s)=\bar u^N(s;t_{N-1},\bar X^{N-1}(t_{N-1})),\end{cases}\qquad s\in[t_{N-1},t_N]. \tag{4.33}$$

Thus, the extended $(\bar X^{N-1}(\cdot),\bar u^{N-1}(\cdot))$ satisfies

$$\begin{cases} d\bar X^{N-1}(s)=b\Big(s,\bar X^{N-1}(s),\psi\big(t_{N-2},s,\bar X^{N-1}(s),V_x^\Pi(s,\bar X^{N-1}(s)),V_{xx}^\Pi(s,\bar X^{N-1}(s))\big)\Big)ds\\ \qquad\qquad +\sigma\Big(s,\bar X^{N-1}(s),\psi\big(t_{N-2},s,\bar X^{N-1}(s),V_x^\Pi(s,\bar X^{N-1}(s)),V_{xx}^\Pi(s,\bar X^{N-1}(s))\big)\Big)dW(s),\\ \qquad\qquad\qquad s\in[t_{N-2},t_{N-1}),\\ d\bar X^{N-1}(s)=b\Big(s,\bar X^{N-1}(s),\psi\big(t_{N-1},s,\bar X^{N-1}(s),V_x^\Pi(s,\bar X^{N-1}(s)),V_{xx}^\Pi(s,\bar X^{N-1}(s))\big)\Big)ds\\ \qquad\qquad +\sigma\Big(s,\bar X^{N-1}(s),\psi\big(t_{N-1},s,\bar X^{N-1}(s),V_x^\Pi(s,\bar X^{N-1}(s)),V_{xx}^\Pi(s,\bar X^{N-1}(s))\big)\Big)dW(s),\\ \qquad\qquad\qquad s\in[t_{N-1},t_N],\\ \bar X^{N-1}(t_{N-2})=x.\end{cases} \tag{4.34}$$

We refer to such a pair $\big(\bar X^{N-1}(\cdot),\bar u^{N-1}(\cdot)\big)$ as a *sophisticated equilibrium pair* on $[t_{N-2},t_N]$. Let

$$\ell^\Pi(s)=\sum_{k=1}^N t_{k-1}I_{[t_{k-1},t_k)}(s),\qquad s\in[0,T]. \tag{4.35}$$

It is easy to see that

$$0\le s-\ell^\Pi(s)\le\|\Pi\|,\qquad s\in[0,T]. \tag{4.36}$$

Then (4.34) can be written compactly as

$$\begin{cases} d\bar X^{N-1}(s)=b\Big(s,\bar X^{N-1}(s),\psi\big(\ell^\Pi(s),s,\bar X^{N-1}(s),V_x^\Pi(s,\bar X^{N-1}(s)),V_{xx}^\Pi(s,\bar X^{N-1}(s))\big)\Big)ds\\ \qquad\qquad +\sigma\Big(s,\bar X^{N-1}(s),\psi\big(\ell^\Pi(s),s,\bar X^{N-1}(s),V_x^\Pi(s,\bar X^{N-1}(s)),V_{xx}^\Pi(s,\bar X^{N-1}(s))\big)\Big)dW(s),\\ \qquad\qquad\qquad s\in[t_{N-2},t_N],\\ \bar X^{N-1}(t_{N-2})=x.\end{cases} \tag{4.37}$$

Also, one has

$$\begin{aligned} J(t_{N-2},x;\bar u^{N-1}(\cdot))&=\mathbb{E}_{t_{N-2}}\Big[\int_{t_{N-1}}^{t_{N-1}}g(t_{N-2},s,\bar X^{N-1}(s),\bar u^{N-1}(s))ds\\ &\qquad+\int_{t_{N-1}}^{t_N}g(t_{N-2},s,\bar X^{N-1}(s),\bar u^{N-1}(s))ds+h(t_{N-2},\bar X^{N-1}(T))\Big]\\ &=J^{N-1}(t_{N-2},x;\bar u^{N-1}(\cdot))=V^\Pi(t_{N-2},x),\qquad x\in\mathbb{R}^n.\end{aligned} \tag{4.38}$$

We point out that in general it may happen that,

$$J\big(t_{N-2},x;\bar u^{N-1}(\cdot)\big)>\inf_{u(\cdot)\in\mathcal{U}[t_{N-2},t_N]}J\big(t_{N-2},x;u(\cdot)\big), \tag{4.39}$$

which means that the the sophisticated equilibrium pair might not be an optimal pair (for the given initial pair).

Similar to the above, in order to state an optimal control problem for Player $(N-2)$ on $[t_{N-3},t_{N-2}]$, we introduce the following BSDE on $[t_{N-2},t_N]$:

$$\begin{cases} dY^{N-1}(s)=-g\big(t_{N-3},s,\bar X^{N-1}(s),\bar u^{N-1}(s)\big)ds+Z^{N-1}(s)dW(s),\quad s\in[t_{N-2},t_N],\\ Y^{N-1}(t_N)=h(t_{N-3},\bar X^{N-1}(t_N)),\end{cases} \tag{4.40}$$

where $\big(\bar{X}^{N-1}(\cdot),\bar{u}^{N-1}(\cdot)\big)$ is the sophisticated equilibrium pair determined by (4.37) on $[t_{N-2},t_N]$, uniquely depending on $x\in\mathbb{R}^n$. Let $\big(Y^{N-1}(\cdot),Z^{N-1}(\cdot)\big)\equiv\big(Y^{N-1}(\cdot\,;x),Z^{N-1}(\cdot\,;x)\big)$ be the adapted solution of this BSDE. Then (4.37) and (4.40) form an FBSDE. Similar to the above, we have

$$Y^{N-1}(s)=\Theta^{N-1}(s,\bar{X}^{N-1}(s)),\qquad s\in[t_{N-2},t_N],\tag{4.41}$$

as long as $\Theta^{N-1}(\cdot\,,\cdot)$ is the solution to the following PDE:

$$\begin{cases}\Theta^{N-1}_s(s,x)+\mathbb{H}\big(t_{N-3},s,x,\psi\big(\ell^\Pi(s),s,x,V^\Pi_x(s,x),V^\Pi_{xx}(s,x)\big),\Theta^{N-1}_x(s,x),\Theta^{N-1}_{xx}(s,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad(s,x)\in[t_{N-2},t_N]\times\mathbb{R}^n,\\ \Theta^{N-1}(t_N,x)=h(t_{N-3},x),\qquad x\in\mathbb{R}^n.\end{cases}\tag{4.42}$$

Having the above preparation, we now consider, for any $(t,x)\in[t_{N-3},t_{N-2})\times\mathbb{R}^n$, the state equation

$$\begin{cases}dX^{N-2}(s)=b(s,X^{N-2}(s),u^{N-2}(s))ds+\sigma(s,X^{N-2}(s),u^{N-2}(s))dW(s),\qquad s\in[t,t_{N-2}],\\ X^{N-2}(t)=x,\end{cases}\tag{4.43}$$

and the (sophisticated) cost functional

$$J^{N-2}(t,x;u^{N-2}(\cdot))=\mathbb{E}_t\Big[\int_t^{t_{N-2}}g(t_{N-3},s,X^{N-2}(s),u^{N-2}(s))ds+\Theta^{N-1}(t_{N-2},X^{N-2}(t_{N-2}))\Big].\tag{4.44}$$

We pose the following problem for Player $(N-2)$:

Problem ($\mathbf{C}_{N-2}$). For any $(t,x)\in[t_{N-3},t_{N-2})\times\mathbb{R}^n$, find a $\bar{u}^{N-2}(\cdot)\equiv\bar{u}^{N-2}(\cdot\,;t,x)\in\mathcal{U}[t_{N-3},t_{N-2}]$ such that

$$J^{N-2}(t,x;\bar{u}^{N-2}(\cdot))=\inf_{u^{N-2}(\cdot)\in\mathcal{U}[t_{N-3},t_{N-2}]}J^{N-2}(t,x;u^{N-2}(\cdot))=V^\Pi(t,x).\tag{4.45}$$

Together with the previous definition, we see that $V^\Pi(\cdot\,,\cdot)$ is now well-defined on $[t_{N-3},t_N]\times\mathbb{R}^n$. Under proper conditions, $V^\Pi(\cdot\,,\cdot)$ is a classical solution to the following HJB equation:

$$\begin{cases}V^\Pi_t(t,x)+\inf\limits_{u\in U}\mathbb{H}\big(\ell^\Pi(t),t,x,u,V^\Pi_x(t,x),V^\Pi_{xx}(t,x)\big)=0,\qquad(t,x)\in[t_{N-3},t_{N-2})\times\mathbb{R}^n,\\ V^\Pi(t_{N-2}-0,x)=\Theta^{N-1}(t_{N-2},x),\qquad x\in\mathbb{R}^n.\end{cases}\tag{4.46}$$

Further, by the definition of the map $\psi(\cdot)$, we may also write the above as

$$\begin{cases}V^\Pi_t(t,x)+\mathbb{H}\big(\ell^\Pi(t),t,x,\psi\big(\ell^\Pi(t),t,x,V^\Pi_x(t,x),V^\Pi_{xx}(t,x)\big),V^\Pi_x(t,x),V^\Pi_{xx}(t,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad(t,x)\in[t_{N-3},t_{N-2})\times\mathbb{R}^n,\\ V^\Pi(t_{N-2}-0,x)=\Theta^{N-1}(t_{N-2},x),\qquad x\in\mathbb{R}^n.\end{cases}\tag{4.47}$$

Also, similar to (4.27), we have that in general,

$$V^\Pi(t_{N-2}-0,x)\neq V^\Pi(t_{N-2},x).\tag{4.48}$$

The above procedure can be continued recursively. By induction, we can construct sophisticated cost functional $J^k(t,x;u^k(\cdot))$ for Player k, and

$$V^\Pi(t,x)=\inf_{u^k(\cdot)\in\mathcal{U}[t,t_k]}J^k(t,x;u^k(\cdot)),\qquad(t,x)\in[t_{k-1},t_k)\times\mathbb{R}^n,\ 1\le k\le N,\tag{4.49}$$

with the value function $V^\Pi(\cdot\,,\cdot)$ satisfying the following HJB equations on the time intervals associated with the partition Π:

$$\begin{cases}V^\Pi_t(t,x)+\mathbb{H}\big(\ell^\Pi(t),t,x,\psi\big(\ell^\Pi(t),t,x,V^\Pi_x(t,x),V^\Pi_{xx}(t,x)\big),V^\Pi_x(t,x),V^\Pi_{xx}(t,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad(t,x)\in[t_{N-1},t_N]\times\mathbb{R}^n,\\ V^\Pi(t_N,x)=h(t_{N-1},x),\qquad x\in\mathbb{R}^n,\end{cases}\tag{4.50}$$

and for $k=1,2,\cdots,N-1$,

$$\begin{cases} V_t^\Pi(t,x)+\mathbb{H}\big(\ell^\Pi(t),t,x,\psi\big(\ell^\Pi(t),t,x,V_x^\Pi(t,x),V_{xx}^\Pi(t,x)\big),V_x^\Pi(t,x),V_{xx}^\Pi(t,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[t_{k-1},t_k)\times\mathbb{R}^n,\\ V^\Pi(t_k-0,x)=\Theta^{k+1}(t_k,x),\qquad x\in\mathbb{R}^n,\end{cases}\tag{4.51}$$

where, for $k=1,2,\cdots,N-1$, $\Theta^{k+1}(\cdot\,,\cdot)$ is the solution to the following (linear) PDE:

$$\begin{cases} \Theta_t^{k+1}(t,x)+\mathbb{H}\big(t_{k-1},t,x,\psi\big(\ell^\Pi(t),t,x,V_x^\Pi(t,x),V_{xx}^\Pi(t,x)\big),\Theta_x^{k+1}(t,x),\Theta_{xx}^{k+1}(t,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[t_k,t_N]\times\mathbb{R}^n,\\ \Theta^{k+1}(t_N,x)=h(t_{k-1},x),\qquad x\in\mathbb{R}^n.\end{cases}\tag{4.52}$$

Now, we define

$$\Psi^\Pi(t,x)=\psi\big(\ell^\Pi(t),t,x,V_x^\Pi(t,x),V_{xx}^\Pi(t,x)\big),\qquad (t,x)\in[0,T]\times\mathbb{R}^n.\tag{4.53}$$

Then for any given $x\in\mathbb{R}^n$, let $\bar X^\Pi(\cdot)$ be the solution to the following closed-loop system:

$$\begin{cases} d\bar X^\Pi(s)=b\big(s,\bar X^\Pi(s),\Psi^\Pi(s,\bar X^\Pi(s))\big)ds+\sigma\big(s,\bar X^\Pi(s),\Psi^\Pi(s,\bar X^\Pi(s))\big)dW(s),\qquad s\in[0,T],\\ \bar X^\Pi(0)=x,\end{cases}\tag{4.54}$$

and denote

$$\bar u^\Pi(s)=\Psi^\Pi\big(s,\bar X^\Pi(s)\big),\qquad s\in[0,T].\tag{4.55}$$

According to our construction, we have

$$\begin{aligned}&J\big(t_{k-1},\bar X^\Pi(t_{k-1});\Psi^\Pi(\cdot)\big|_{[t_{k-1},T]}\big)=J\big(t_{k-1},\bar X^\Pi(t_{k-1});\bar u^\Pi(\cdot)\big|_{[t_{k-1},t_k]}\big)=V^\Pi\big(t_{k-1},\bar X^\Pi(t_{k-1})\big)\\ &=J^k\big(t_{k-1},\bar X^\Pi(t_{k-1});\bar u^\Pi(\cdot)\big|_{[t_{k-1},t_k]}\big)\\ &=\inf_{u^k(\cdot)\in\mathcal{U}[t_{k-1},t_k]}J^k\big(t_{k-1},\bar X^\Pi(t_{k-1});u^k(\cdot)\big)\le J^k\big(t_{k-1},\bar X^\Pi(t_{k-1});u^k(\cdot)\big)\\ &=J\big(t_{k-1},\bar X^\Pi(t_{k-1});u^k(\cdot)\oplus\Psi^\Pi(\cdot)\big|_{[t_k,T]}\big),\qquad\forall u^k(\cdot)\in\mathcal{U}[t_{k-1},t_k],\quad 1\le k\le N,\end{aligned}\tag{4.56}$$

where $u^k(\cdot)\oplus\Psi^\Pi(\cdot)\big|_{[t_k,T]}$ is defined the same way as (4.8)–(4.9). Similar to (4.39), for $k=1,2,\cdots,N-1$, we have in general that

$$J\big(t_{k-1},\bar X^\Pi(t_{k-1});\bar u^\Pi(\cdot)\big)>\inf_{u(\cdot)\in\mathcal{U}[t_{k-1},T]}J\big(t_{k-1},\bar X^\Pi(t_{k-1});u(\cdot)\big).\tag{4.57}$$

Since the involved N players in Problem (G^Π) interact through the initial/terminal pairs $(t_k,X(t_k))$, $k=1,2,\cdots,N-1$, one should actually denote

$$J^k\big(t_{k-1},X(t_{k-1});u^k(\cdot)\big)\equiv\widetilde J^k\big(x;u^1(\cdot),\cdots,u^N(\cdot)\big),\qquad 1\le k\le N.\tag{4.58}$$

Hence, (4.56) means that if we let

$$\bar u^k(\cdot)=\bar u^\Pi(\cdot)\big|_{[t_{k-1},t_k]},\qquad 1\le k\le N,$$

then $(\bar u^1(\cdot),\cdots,\bar u^N(\cdot))$ is a *Nash equilibrium* of the N-person non-cooperative differential game associated with $\widetilde J^k\big(x;u^1(\cdot),\cdots,u^N(\cdot)\big)$, $1\le k\le N$ (defined in (4.58)).

4.2 The formal limits.

We now would like to look at the situation when $\|\Pi\| \to 0$. Suppose we have the following:

$$\lim_{\|\Pi\|\to 0}\Big(|V^\Pi(t,x)-V(t,x)|+|V_x^\Pi(t,x)-V_x(t,x)|+|V_{xx}^\Pi(t,x)-V_{xx}(t,x)|\Big)=0, \tag{4.59}$$

uniformly for (t,x) in any compact sets, for some $V(\cdot\,,\cdot)$. Under (H3), we also have

$$\lim_{\|\Pi\|\to 0}|\Psi^\Pi(t,x)-\Psi(t,x)|=0, \tag{4.60}$$

uniformly for (t,x) in any compact sets, for

$$\Psi(t,x)=\psi\big(t,t,x,V_x(t,x),V_{xx}(t,x)\big),\qquad (t,x)\in[0,T]\times\mathbb{R}^n. \tag{4.61}$$

Then the following limit exist:

$$\lim_{\|\Pi\|\to 0}\|\bar X^\Pi(\cdot)-\bar X(\cdot)\|_{L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))}=0,$$

for $\bar X(\cdot)$ solving the following SDE:

$$\begin{cases} d\bar X(s)=b\big(s,\bar X(s),\bar u(s)\big)ds+\sigma\big(s,\bar X(s),\bar u(s)\big)dW(s),\quad s\in[0,T]. \\ \bar X(0)=x, \end{cases} \tag{4.62}$$

where

$$\bar u(s)=\Psi\big(s,\bar X(s)\big),\qquad s\in[0,T], \tag{4.63}$$

and

$$L^2_{\mathbb{F}}(\Omega,C([0,T];\mathbb{R}^n))=\Big\{X:[0,T]\times\Omega\to\mathbb{R}^n\;\big|\;X(\cdot)\text{ has continuous paths,}\\ \mathbb{E}\big[\sup_{t\in[0,T]}|X(t)|^2\big]<\infty\Big\}.$$

Clearly,

$$\lim_{\|\Pi\|\to 0}\|\bar u^\Pi(\cdot)-\bar u(\cdot)\|_{\mathcal{U}^2[0,T]}=0. \tag{4.64}$$

By (4.56), we have

$$J\big(\ell^\Pi(t),\bar X^\Pi(\ell^\Pi(t));\bar u^\Pi(\cdot)\big)=V^\Pi(\ell^\Pi(t),\bar X^\Pi(t)),\qquad t\in[0,T].$$

Thus, passing to the limits, we have (4.5). Also, we have the following:

$$\begin{aligned}
&J\big(t_{k-1},\bar X(t_{k-1});\Psi(\cdot)\big|_{[t_{k-1},T]}\big)\equiv J\big(t_{k-1},\bar X(t_{k-1});\bar u(\cdot)\big|_{[t_{k-1},T]}\big)\\
&=\mathbb{E}_{t_{k-1}}\Big[\int_{t_{k-1}}^T g(t_{k-1},s,\bar X(s),\bar u(s))ds+h(t_{k-1},\bar X(T))\Big]\\
&\le\mathbb{E}_{t_{k-1}}\Big[\int_{t_{k-1}}^T g(t_{k-1},s,\bar X^\Pi(s),\bar u^\Pi(s))ds+h(t_{k-1},\bar X^\Pi(T))\Big]+R\big(\|\Pi\|\big)\\
&=V^\Pi\big(t_{k-1},\bar X^\Pi(t_{k-1})\big)+R\big(\|\Pi\|\big)=J^k\big(t_{k-1},\bar X^\Pi(t_{k-1});\bar u^\Pi(\cdot)\big|_{[t_{k-1},T]}\big)+R\big(\|\Pi\|\big)\\
&\le J^k\big(t_{k-1},\bar X^\Pi(t_{k-1}),u^k(\cdot)\big)+R\big(\|\Pi\|\big)\\
&=J\big(t_{k-1},\bar X(t_{k-1});u^k(\cdot)\oplus\Psi(\cdot)\big|_{[t_k,T]}\big)+R\big(\|\Pi\|\big),\qquad\forall u^k(\cdot)\in\mathcal{U}[t_{k-1},t_k],
\end{aligned}$$

for some $R(r)$ with $R(r)\to 0$ as $r\to 0$. Hence, by Definition 4.1, $\Psi(\cdot\,,\cdot)$ is a time-consistent equilibrium strategy, and $V(\cdot\,,\cdot)$ is a time-consistent equilibrium value function of Problem (N).

In the rest of this subsection, we will formally pass to the limits to find the equations that can be used to characterize the equilibrium value function $V(\cdot\,,\cdot)$. To this end, let us first write the equations for $\Theta^{k+1}(\cdot\,,\cdot)$ in the integral forms: For $k=1,2,\cdots,N-1$, one has

$$\begin{aligned}\Theta^{k+1}(t,x)&=h(t_{k-1},x)+\int_t^T\mathbb{H}\big(t_{k-1},s,x,\psi\big(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x)\big),\Theta_x^{k+1}(s,x),\Theta_{xx}^{k+1}(s,x)\big)ds\\&=h(t_{k-1},x)+\int_t^T\Big(\langle\,\Theta_x^{k+1}(s,x),b\big(s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\,\rangle\\&\qquad+\mathrm{tr}\,\big[a\big(s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\Theta_{xx}^{k+1}(s,x)\big]\\&\qquad+g\big(t_{k-1},s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\Big)ds,\\&\qquad\qquad(t,x)\in[t_k,T]\times\mathbb{R}^n.\end{aligned}\tag{4.65}$$

Let us define

$$\Theta^\Pi(\tau,t,x)=\sum_{k=1}^{N-1}\Theta^{k+1}(t,x)I_{[t_{k-1},t_k)}(\tau),\qquad(\tau,t,x)\in D[0,T]\times\mathbb{R}^n.\tag{4.66}$$

Then

$$\begin{aligned}\Theta^\Pi(\tau,t,x)=h^\Pi(\tau,x)&+\int_t^T\Big\{\langle\,\Theta_x^\Pi(\tau,s,x),b\big(s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\,\rangle\\&+\mathrm{tr}\,\big[a\big(s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\Theta_{xx}^\Pi(\tau,s,x)\big]\\&+g^\Pi\big(\tau,s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\Big\}ds,\end{aligned}\tag{4.67}$$

where

$$\begin{cases}h^\Pi(\tau,x)=\displaystyle\sum_{k=1}^{N-1}h(t_{k-1},x)I_{[t_{k-1},t_k)}(\tau),\qquad(\tau,x)\in[0,T]\times\mathbb{R}^n,\\ g^\Pi(\tau,s,x,u)=\displaystyle\sum_{k=1}^{N-1}g(t_{k-1},s,x,u)I_{[t_{k-1},t_k)}(\tau),\qquad(\tau,s,x,u)\in D[0,T]\times\mathbb{R}^n\times U.\end{cases}$$

Let us look at $V^\Pi(\cdot\,,\cdot)$. For $(t,x)\in[t_{N-1},t_N]\times\mathbb{R}^n$, we have

$$\begin{aligned}V^\Pi(t,x)&=h(t_{N-1},x)+\int_t^T\mathbb{H}\big(t_{N-1},s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x)),V_x^\Pi(s,x),V_{xx}^\Pi(s,x)\big)ds\\&=h(\ell^\Pi(t),x)+\int_t^T\mathbb{H}\big(\ell^\Pi(t),s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x)),V_x^\Pi(s,x),V_{xx}^\Pi(s,x)\big)ds,\end{aligned}\tag{4.68}$$

and for $(t,x)\in[t_{k-1},t_k)\times\mathbb{R}^n$, $k=1,2,\cdots,N-1$,

$$\begin{aligned}V^\Pi(t,x)&=\Theta^{k+1}(t_k,x)+\int_t^{t_k}\mathbb{H}\Big(\ell^\Pi(s),s,x,\psi\big(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x)\big),V_x^\Pi(s,x),V_{xx}^\Pi(s,x)\Big)ds\\&=\Theta^\Pi(t_{k-1},t_k,x)+\int_t^{t_k}\Big\{\langle\,V_x^\Pi(s,x),b\big(s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\,\rangle\\&\qquad+\mathrm{tr}\,\big[a\big(s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)V_{xx}^\Pi(s,x)\big]\\&\qquad+g^\Pi\big(t_{k-1},s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\Big\}ds.\end{aligned}$$

On the other hand, for $(t,x)\in[t_{k-1},t_k)\times\mathbb{R}^n$, we have

$$\begin{aligned}\Theta^\Pi(t_{k-1},t,x)=\Theta^\Pi(t_{k-1},t_k,x)&+\int_t^{t_k}\Big\{\langle\,\Theta_x^\Pi(t_{k-1},s,x),b\big(s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\,\rangle\\&+\mathrm{tr}\,\big[a\big(s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\Theta_{xx}^\Pi(t_{k-1},s,x)\big]\\&+g^\Pi\big(t_{k-1},s,x,\psi(\ell^\Pi(s),s,x,V_x^\Pi(s,x),V_{xx}^\Pi(s,x))\big)\Big\}ds.\end{aligned}$$

Therefore, in the case that (4.67) is well-posed, we have

$$V^{\Pi}(t,x)=\Theta^{\Pi}(t_{k-1},t,x),\qquad (t,x)\in[t_{k-1},t_k)\times\mathbb{R}^n,\tag{4.69}$$

or equivalently,

$$V^{\Pi}(t,x)=\Theta^{\Pi}(\ell^{\Pi}(t),t,x),\qquad (t,x)\in[0,t_{N-1})\times\mathbb{R}^n.\tag{4.70}$$

Now, let us assume (4.59) holds for some $V(\cdot\,,\cdot)$ and

$$\begin{aligned}\lim_{\|\Pi\|\to0}\Big(|\Theta^{\Pi}(\tau,t,x)-\Theta(\tau,t,x)|+|\Theta_x(\tau,t,x)-\Theta_x(\tau,t,x)|+|\Theta_{xx}(\tau,t,x)-\Theta_{xx}(\tau,t,x)|\Big)=0,\\ (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\end{aligned}\tag{4.71}$$

for some $\Theta(\cdot\,,\cdot\,,\cdot)$. Let us assume that

$$|h_\tau(\tau,x)|+|g_\tau(\tau,t,x,u)|\le K,\qquad \forall(\tau,t,x,u)\in D[0,T]\times\mathbb{R}^n\times U.\tag{4.72}$$

Then we have

$$\lim_{\|\Pi\|\to0}h^{\Pi}(\tau,x)=h(\tau,x),\qquad (\tau,x)\in[0,T]\times\mathbb{R}^n,$$

and

$$\begin{aligned}\lim_{\|\Pi\|\to0}g^{\Pi}\big(\tau,s,x,\psi(\ell^{\Pi}(s),s,x,V_x^{\Pi}(s,x),V_{xx}^{\Pi}(s,x))\big)=g\big(\tau,s,x,\psi(s,s,x,V_x(s,x),V_{xx}(s,x))\big),\\ (\tau,s,x)\in D[0,T]\times\mathbb{R}^n.\end{aligned}$$

Consequently, we obtain the following integro-partial differential equation for $\Theta(\cdot\,,\cdot\,,\cdot)$:

$$\begin{aligned}\Theta(\tau,t,x)=h(\tau,x)+\int_t^T\mathbb{H}\big(\tau,s,x,\psi(s,s,x,\Theta_x(s,s,x),\Theta_{xx}(s,s,x)),\Theta_x(\tau,s,x),\Theta_{xx}(\tau,s,x)\big)ds,\\ (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\end{aligned}\tag{4.73}$$

and relation

$$V(t,x)=\Theta(t,t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^n.\tag{4.74}$$

It is clear that (4.73) is an integral form of the following differential equation:

$$\begin{cases}\Theta_t(\tau,t,x)+\mathbb{H}\big(\tau,t,x,\psi\big(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x)\big),\Theta_x(\tau,t,x),\Theta_{xx}(\tau,t,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x),\qquad (\tau,x)\in[0,T]\times\mathbb{R}^n,\end{cases}\tag{4.75}$$

where

$$\begin{cases}\mathbb{H}(\tau,t,x,u,p,P)=\operatorname{tr}\big[a(t,x,u)P\big]+\langle\,b(t,x,u),p\,\rangle+g(\tau,t,x,u),\\ \psi(\tau,t,x,p,P)\in\arg\min\mathbb{H}(\tau,t,x,\cdot\,,p,P).\end{cases}\tag{4.76}$$

Therefore, we may equivalently write (4.75) as follows:

$$\begin{cases}\Theta_t(\tau,t,x)+\langle\,\Theta_x(\tau,t,x),b\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big)\,\rangle\\ \qquad+\operatorname{tr}\big[a\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big)\Theta_{xx}(\tau,t,x)\big]\\ \qquad+g\big(\tau,t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big)=0,\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x),\qquad (\tau,x)\in[0,T]\times\mathbb{R}^n.\end{cases}\tag{4.77}$$

We call the above (4.77) the *equilibrium Hamilton-Jacobi-Bellamn equation* (equilibrium HJB equation, for short) of Problem (N). If one can find $\Theta(\cdot\,,\cdot\,,\cdot)$ from the above, then the equilibrium value function $V(\cdot\,,\cdot)$

can be determined by (4.74), and the (time-consistent) equilibrium pair can be determined by (4.62) and (4.63), in principle.

Let us make some remarks on (4.77).

(i) It is an interesting feature of (4.77) that both $\Theta(\tau,t,x)$ and $\Theta(t,t,x)$ appear in the equation where the later is the restriction of the former on $\tau=t$. On one hand, although the equation is fully nonlinear, due to the fact that $\Theta(t,t,x)$ is different from $\Theta(\tau,t,x)$, the existing theory for fully nonlinear parabolic equations cannot apply directly. On the other hand, it is seen that if $\Theta(t,t,x)$ is obtained from an independent way, then (4.77) is actually a linear equation for $\Theta(\tau,t,x)$ with τ can be purely regarded as a parameter.

(ii) In the case that $\mathcal{D}(\psi)$ is not equal to $D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n$, the condition

$$(\tau,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\in\mathcal{D}(\psi) \tag{4.78}$$

has to be regarded as a part of the solution. We will see that for some interesting special cases, the above condition can come automatically. More generally, we may also write (4.75) as

$$\begin{cases}\Theta_t(\tau,t,x)+\mathbb{H}\big(\tau,t,x,\arg\min\mathbb{H}\big(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x)\big),\Theta_x(\tau,t,x),\Theta_{xx}(\tau,t,x)\big)\ni 0,\\ \qquad\qquad\qquad\qquad\qquad\qquad(\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x),\qquad(\tau,x)\in[0,T]\times\mathbb{R}^n,\end{cases} \tag{4.79}$$

since the set $\arg\min\mathbb{H}\big(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x)\big)$ might contain more than one point.

(iii) In the case $\mathcal{D}(\psi)\neq\mathcal{D}(H)$, according to Lemma 3.2, we can define

$$\mathbb{H}^\varepsilon(\tau,t,x,u,p,P)=\mathbb{H}(\tau,t,x,u,p,P)+\varepsilon|u|^2,$$

and there exists a $\psi^\varepsilon:D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n\to U$ such that

$$\mathbb{H}^\varepsilon(\tau,t,x,\psi^\varepsilon(\tau,t,x,p,P),p,P)=\inf_{u\in U}\mathbb{H}^\varepsilon(\tau,t,x,u,p,P)\equiv H^\varepsilon(\tau,t,x,p,P).$$

Further,

$$\lim_{\varepsilon\to0}H^\varepsilon(\tau,t,x,p,P)=H(\tau,t,x,p,P),\qquad\lim_{\varepsilon\to0}\varepsilon|\psi^\varepsilon(\tau,t,x,p,P)|^2=0.$$

It is not hard to see that the above actually amounts to defining

$$g^\varepsilon(\tau,t,x,u)=g(\tau,t,x,u)+\varepsilon|u|^2.$$

We may refer to the corresponding problem as a *regularized* problem. If the corresponding equilibrium value function is denoted by $V^\varepsilon(\cdot\,,\cdot)$, then it is expected that

$$\lim_{\varepsilon\downarrow0}V^\varepsilon(t,x)=V(t,x),\qquad(t,x)\in[0,T]\times\mathbb{R}^n. \tag{4.80}$$

However, in general, if $(\bar X^\varepsilon(\cdot),\bar u^\varepsilon(\cdot))$ is an equilibrium pair for the regularized problem, we might not have the limit of $\bar u^\varepsilon(\cdot)$ as $\varepsilon\to0$. In this case, we should be satisfied by the above characterization of the equilibrium value function $V(\cdot\,,\cdot)$, and $\bar u^\varepsilon(\cdot)$ can be regarded as some kind of "near equilibrium control".

(iv) For the case

$$\sigma(t,x,u)=\sigma(t,x),\qquad\forall(t,x,u)\in[0,T]\times\mathbb{R}^n\times U, \tag{4.81}$$

i.e., the control does not enter the diffusion of the state equation, $\psi(\cdot)$ is independent of P and

$$\psi(t,x,p)\in\arg\min\big[\,\langle\,p,b(t,x,\cdot)\,\rangle+g(\tau,t,x,\cdot)\big]. \tag{4.82}$$

Then the equilibrium HJB equation can be written as

$$\begin{cases} \Theta_t(\tau,t,x)+\mathrm{tr}\,\big[a(t,x)\Theta_{xx}(\tau,t,x)\big]+\langle\, b\big(t,x,\psi(t,t,x,\Theta_x(t,t,x))\big),\Theta_x(\tau,t,x)\,\rangle \\ \qquad\qquad +g\big(\tau,t,x,\psi(t,t,x,\Theta_x(t,t,x))\big)=0, \qquad\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n, \\ \Theta(\tau,T,x)=h(\tau,x), \qquad (\tau,x)\in[0,T]\times\mathbb{R}^n. \end{cases} \tag{4.83}$$

We will carefully discuss this case in the next section. Note that for a deterministic problem, namely Problem (N) for an ordinary differential equation system, we may take

$$\sigma(t,x,u)=\varepsilon I, \qquad (t,x,u)\in[0,T]\times\mathbb{R}^n\times U,$$

for $\varepsilon>0$ to regularize the problem. Then the corresponding equilibrium HJB equation reads

$$\begin{cases} \Theta^\varepsilon_t(\tau,t,x)+\frac{1}{2}\Delta\Theta^\varepsilon_{xx}(\tau,t,x)+\langle\, b\big(t,x,\psi(t,t,x,\Theta^\varepsilon_x(t,t,x))\big),\Theta^\varepsilon_x(\tau,t,x)\,\rangle \\ \qquad\qquad +g\big(\tau,t,x,\psi(t,t,x,\Theta^\varepsilon_x(t,t,x))\big)=0, \qquad\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n, \\ \Theta^\varepsilon(\tau,T,x)=h(\tau,x), \qquad (\tau,x)\in[0,T]\times\mathbb{R}^n. \end{cases} \tag{4.84}$$

It is expected that $\Theta^\varepsilon(\cdot\,,\cdot\,,\cdot)\to\Theta(\cdot\,,\cdot\,,\cdot)$ in some sense, as $\varepsilon\to0$, with

$$\begin{cases} \Theta_t(\tau,t,x)+\langle\, b\big(t,x,\psi(t,t,x,\Theta_x(t,t,x))\big),\Theta_x(\tau,t,x)\,\rangle+g\big(\tau,t,x,\psi(t,t,x,\Theta_x(t,t,x))\big)=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n, \\ \Theta(\tau,T,x)=h(\tau,x), \qquad (\tau,x)\in[0,T]\times\mathbb{R}^n. \end{cases} \tag{4.85}$$

At the moment, it is not clear to us how one can define viscosity solution to the above equation.

5 Well-Posedness of the Equilibrium HJB Equation

In this section, we discuss the well-posedness for the equilibrium HJB equation (4.77). Let us first intuitively describe our idea. For any smooth function $v(\cdot\,,\cdot)$, denote

$$\begin{aligned} \big[\mathcal{L}\big(t,v(t,\cdot)\big)\varphi(\cdot)\big](x)=\mathrm{tr}\,\big[&a(t,x,\psi(t,t,x,v_x(t,x),v_{xx}(t,x))\varphi_{xx}(x)\big] \\ &+\langle\, b\big(t,x,\psi(t,t,x,v_x(t,x))\big),\varphi_x(x)\,\rangle, \qquad (t,x)\in[0,T]\times\mathbb{R}^n, \end{aligned} \tag{5.1}$$

and

$$\mathcal{G}(\tau,t,v(t,\cdot))(x)=g\big(\tau,t,x,\psi(t,t,x,v_x(t,x))\big), \qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n. \tag{5.2}$$

Consider the following linear abstract backward evolution equation:

$$\begin{cases} \Theta_t(\tau,t)+\mathcal{L}(t,v(t))\Theta(\tau,t)+\mathcal{G}(\tau,t,v(t))=0, \qquad t\in[\tau,T], \\ \Theta(\tau,T)=h(\tau). \end{cases} \tag{5.3}$$

Under some mild conditions, the above is well-posed, and we have the following variation of constant formula:

$$\Theta(\tau,t)=\mathcal{E}(T,t;v(\cdot))h(\tau)+\int_t^T\mathcal{E}(s,t;v(\cdot))\mathcal{G}(\tau,s,v(s))ds, \qquad t\in[\tau,T], \tag{5.4}$$

where $\mathcal{E}(\cdot\,,\cdot\,;v(\cdot))$ is called the *backward evolution operator* generated by $\mathcal{L}(\cdot\,,v(\cdot))$. Consequently, the (time-consistent) equilibrium value function $V(t,\cdot)=\Theta(t,t,\cdot)$ should be the solution to the following nonlinear functional integral equation:

$$V(t)=\mathcal{E}(T,t;V(\cdot))h(t)+\int_t^T\mathcal{E}(s,t;V(\cdot))\mathcal{G}(t,s,V(s))ds, \qquad t\in[0,T]. \tag{5.5}$$

We call (5.5) the *equilibrium HJB integral equation* for Problem (N). Once a solution $V(\cdot\,,\cdot)$ of (5.5) is found, we can, in principle, construct a (time-consistent) equilibrium control and an equilibrium pair for Problem (N). Of course, if we like, we may also solve the equilibrium HJB equation (4.77), which actually is not necessary as far as the construction of a time-consistent equilibrium pair is concerned.

The well-posedness of (5.5) seems to be difficult for the general case. At the moment, we do not have a complete solution for that and hopefully, we can present some satisfactory results for the equilibrium HJB integral equation (5.5) in our future publications. On the other hand, in the rest of this section, we are going to present a well-posedness result for an interesting special case of (5.5), from which one can get some taste of the problem. The main hypothesis that we will assume below is the following:

$$\sigma(t,x,u)=\sigma(t,x),\qquad (t,x,u)\in[0,T]\times\mathbb{R}^n\times U,\tag{5.6}$$

namely, the control does not enter the diffusion of the state equation. As we discussed in Section 4, in this case, our equilibrium HJB equation reads

$$\begin{cases}\Theta_t(\tau,t,x)+\frac{1}{2}\mathrm{tr}\left[\sigma(t,x)\sigma(t,x)^T\Theta_{xx}(\tau,t,x)\right]+\langle\, b\big(t,x,\psi(t,t,x,\Theta_x(t,t,x))\big),\Theta_x(\tau,t,x)\,\rangle\\ \qquad\qquad +g\big(\tau,t,x,\psi(t,t,x,\Theta_x(t,t,x))\big)=0,\qquad\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x),\qquad (\tau,x)\in[0,T]\times\mathbb{R}^n.\end{cases}\tag{5.7}$$

The essential feature of (5.7) is that $\Theta_{xx}(t,t,x)$ does not appear in the equation (although $\Theta_x(t,t,x)$ still appears there). This leads to the well-posedness problem much more accessible. Further, from Example 3.5, we see that there are cases for which ψ is as smooth as the coefficients and $b(t,x,\psi(t,t,x,p))$ is bounded. Therefore, the case that we are going to consider below, although very special, includes a big class of problems.

To avoid heavy notations, let us consider the following equation

$$\begin{cases}\Theta_t(\tau,t,x)+\mathrm{tr}\left[a(t,x)\Theta_{xx}(\tau,t,x)\right]+\langle\, b\big(t,x,\Theta_x(t,t,x)\big),\Theta_x(\tau,t,x)\,\rangle+g\big(\tau,t,x,\Theta_x(t,t,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x),\qquad (\tau,x)\in[0,T]\times\mathbb{R}^n,\end{cases}\tag{5.8}$$

with

$$\begin{cases}a(t,x)=\frac{1}{2}\sigma(t,x)\sigma(t,x)^T,\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\\ b(t,x,p)=b(t,x,\psi(t,t,x,p)),\qquad (t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n,\\ g(\tau,t,x,p)=g(\tau,t,x,\psi(t,t,x,p)),\qquad (\tau,t,x,p)\in D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n.\end{cases}\tag{5.9}$$

To investigate the well-posedness of (5.8) above, let us make some preparations. Let $C^\alpha(\mathbb{R}^n)$ be the space of all continuous functions $\varphi:\mathbb{R}^n\to\mathbb{R}$ such that

$$\|\varphi\|_\alpha\equiv\|\varphi\|_0+[\varphi]_\alpha<\infty,$$

where

$$\|\varphi\|_0=\sup_{x\in\mathbb{R}^n}|\varphi(x)|,\quad [\varphi]_\alpha=\sup_{x,y\in\mathbb{R}^n,x\neq y}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}.$$

Further, let $C^{1+\alpha}(\mathbb{R}^n)$ and $C^{2+\alpha}(\mathbb{R}^n)$ be the spaces of all functions $\varphi:\mathbb{R}^n\to\mathbb{R}$ such that

$$\|\varphi\|_{1+\alpha}\equiv\|\varphi\|_0+\|\varphi_x\|_0+[\varphi_x]_\alpha<\infty,$$

and

$$\|\varphi\|_{2+\alpha}\equiv\|\varphi\|_0+\|\varphi_x\|_0+\|\varphi_{xx}\|_0+[\varphi_{xx}]_\alpha<\infty,$$

respectively. Next, let $B([0,T];C^\alpha(\mathbb{R}^n))$ be the set of all measurable functions $f:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ such that for each $t\in[0,T]$, $f(t,\cdot)\in C^\alpha(\mathbb{R}^n)$ and

$$\|f(\cdot\,,\cdot)\|_{B([0,T];C^\alpha(\mathbb{R}^n))}=\sup_{t\in[0,T]}\|f(t,\cdot)\|_\alpha<\infty.$$

Also, we let $C([0,T];C^\alpha(\mathbb{R}^n))$ be the set of all continuous functions that are also in $B([0,T];C^\alpha(\mathbb{R}^n))$. Thus,

$$C([0,T];C^\alpha(\mathbb{R}^n))\subseteq B([0,T];C^\alpha(\mathbb{R}^n)).$$

Similarly, we define $B([0,T];C^{k+\alpha}(\mathbb{R}^n))$ and $C([0,T];C^{k+\alpha}(\mathbb{R}^n))$, respectively, for $k=1,2$.

We introduce the following hypotheses for the above equation (5.8).

(P) The maps $a:[0,T]\times\mathbb{R}^n\to\mathcal{S}^n$, $b:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $g:D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and $h:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ are continuous and bounded. Moreover, there exists a constant $L>0$ such that

$$\begin{aligned}|a_x(t,x)|+|b_x(t,x,p)|+|g_x(\tau,t,x,p)|+|b_p(t,x,p)|+|g_p(\tau,t,x,p)|+|h_x(\tau,x)|\le L,\\ (\tau,t,x,p)\in D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n.\end{aligned}\tag{5.10}$$

Further, $a(t,x)^{-1}$ exists for all $(t,x)\in[0,T]\times\mathbb{R}^n$ and there exist constants $\lambda_0,\lambda_1>0$ such that

$$\lambda_0 I\le a(t,x)^{-1}\le\lambda_1 I,\qquad\forall(t,x)\in[0,T]\times\mathbb{R}^n.\tag{5.11}$$

We point out here that some of the conditions assumed in (P) can be substantially relaxed. However, we prefer not to get into those generalities for the sake of simplicity in our presentation. Note also that typically, the ellipticity condition of $a(t,x)$ looks like

$$a(t,x)\ge\delta I,\qquad\forall(t,x)\in[0,T]\times\mathbb{R}^n,$$

for some $\delta>0$. It is clear that when $a(\cdot\,,\cdot)$ is assumed to be bounded, then the above is equivalent to (5.11). The number λ_0 in (5.11) will be used below.

For any $v(\cdot\,,\cdot)\in C([0,T];C^{1+\alpha}(\mathbb{R}^n))$, we consider the following linear PDE, parameterized by $\tau\in[0,T)$:

$$\left\{\begin{array}{ll}\Theta_t(\tau,t,x)+\mathcal{L}[t,v(\cdot)]\Theta(\tau,t,x)+g\big(\tau,t,x,v_x(t,x)\big)=0, & (t,x)\in[0,T)\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x), & x\in\mathbb{R}^n,\end{array}\right.\tag{5.12}$$

where the differential operator $\mathcal{L}[t,v(\cdot)]$ is defined by the following:

$$\mathcal{L}[t,v(\cdot)]\varphi(x)=\operatorname{tr}\big[a(t,x)\varphi_{xx}(x)\big]+\langle\, b(t,x,v_x(t,x)),\varphi_x(x)\,\rangle,\qquad\forall\varphi(\cdot)\in C^2(\mathbb{R}^n).\tag{5.13}$$

In what follows, we let

$$C_\lambda(\mathbb{R}^n)=\Big\{\varphi(\cdot)\in C(\mathbb{R}^n)\;\Big|\;\sup_{x\in\mathbb{R}^n}e^{-\lambda|x|^2}|\varphi(x)|<\infty\Big\}.\tag{5.14}$$

We have the following result whose proof follows a relevant one found in [9], with some minor modifications.

Proposition 5.1. *Let* (P) *hold and fix a* $\tau\in[0,T)$. *Then for any* $v(\cdot\,,\cdot)\in C([0,T];C^{1+\alpha}(\mathbb{R}^n))$ *and any* $h(\tau,\cdot)\in C_\lambda(\mathbb{R}^n)$ *with* $\lambda<\frac{\lambda_0}{4T}$, *Cauchy problem* (5.12) *admits a unique classical solution* $\Theta(\tau,\cdot\,,\cdot)$ *and the following representation holds:*

$$\begin{aligned}\Theta(\tau,t,x)=\int_{\mathbb{R}^n}\Gamma^{v(\cdot)}(t,x;T,y)h(\tau,y)dy+\int_t^T\int_{\mathbb{R}^n}\Gamma^{v(\cdot)}(t,x;s,y)g(\tau,s,y,v_x(s,y))dyds,\\ (t,x)\in[0,T]\times\mathbb{R}^n.\end{aligned}\tag{5.15}$$

Here, $\Gamma^{v(\cdot)}(t,x;s,y)$ is defined on $[0,T]\times\mathbb{R}^n\times[0,T]\times\mathbb{R}^n$ with $t<s$ having the following properties:

(i) *For any fixed $(s,y)\in(0,T]\times\mathbb{R}^n$,*

$$\Gamma_x^{v(\cdot)}(t,x;s,y),\ \Gamma_{xx}^{v(\cdot)}(t,x;s,y),\ \Gamma_t^{v(\cdot)}(t,x;s,y)$$

are continuous in $(t,x),(s,y)\in[0,T]\times\mathbb{R}^n$ with $t<s$, and for any fixed $(s,y)\in(0,T]\times\mathbb{R}^n$,

$$\mathcal{L}[t,v(\cdot)]\Gamma^{v(\cdot)}(t,x;s,y)=0,\qquad (t,x)\in[0,s)\times\mathbb{R}^n, \tag{5.16}$$

(ii) *For any $\varphi(\cdot)\in C_\lambda(\mathbb{R}^n)$ with $\lambda<\frac{\lambda_0}{4T}$,*

$$\lim_{t\uparrow s}\int_{\mathbb{R}^n}\Gamma^{v(\cdot)}(t,x;s,y)\varphi(y)dy=\varphi(x),\qquad x\in\mathbb{R}^n, \tag{5.17}$$

The map $\Gamma^{v(\cdot)}(t,x;s,y)$ in the above proposition is called the *fundamental solution* to the problem (5.12).

Our main result of this section is the following.

Theorem 5.2. *Let* (P) *hold. Then* (5.8) *admits a unique solution* $\Theta(\cdot\,,\cdot\,,\cdot)$.

Proof. Let

$$\mathcal{L}_0(t)\varphi(x)=\mathrm{tr}\,\big[a(t,x)\varphi_{xx}(x)\big],\qquad \forall\varphi(\cdot)\in C^2(\mathbb{R}^n), \tag{5.18}$$

which is independent of $v(\cdot)$. Then

$$\mathcal{L}[t,v(\cdot)]\varphi(x)=\mathcal{L}_0(t)\varphi(s)+\langle\, b(t,x,v_x(t,x)),\varphi_x(x)\,\rangle,\qquad \forall\varphi(\cdot)\in C^2(\mathbb{R}^n), \tag{5.19}$$

and (5.12) can be written as (fix $\tau\in[0,T)$)

$$\begin{cases}\Theta_t(\tau,t,x)+\mathcal{L}_0(t)\Theta(\tau,t,x)+\langle\, b(t,x,v_x(t,x)),\Theta_x(\tau,t,x)\,\rangle+g\big(\tau,t,x,v_x(t,x)\big)=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[0,T)\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x),\qquad\qquad x\in\mathbb{R}^n,\end{cases} \tag{5.20}$$

Applying Proposition 5.1, we have

$$\begin{aligned}\Theta(\tau,t,x)=\int_{\mathbb{R}^n}\Gamma^0(t,x;T,y)h(\tau,y)dy+\int_t^T\!\!\int_{\mathbb{R}^n}\Gamma^0(t,x;s,y)\big[\,\langle\, b(s,y,v_x(s,y)),\Theta_x(\tau,s,y)\,\rangle\\ +g(\tau,s,y,v_x(s,y))\big]dyds,\qquad (t,x)\in[\tau,T]\times\mathbb{R}^n,\end{aligned} \tag{5.21}$$

where $\Gamma^0(t,x;s,y)$ is the fundamental solution of $\mathcal{L}_0(\cdot)$, given by the following explicitly:

$$\begin{aligned}\Gamma^0(t,x;s,y)=\frac{1}{\big(4\pi(s-t)\big)^{\frac n2}\big(\det[a(s,y)]\big)^{\frac12}}\,e^{-\frac{\langle\, a(s,y)^{-1}(x-y),(x-y)\,\rangle}{4(s-t)}},\\ (t,x),(s,y)\in[0,T]\times\mathbb{R}^n,\ t<s.\end{aligned} \tag{5.22}$$

Direct computations show that ([9])

$$\begin{cases}|\Gamma^0(t,x;s,y)|\le\dfrac{K}{(s-t)^{\frac n2}}\,e^{-\frac{\lambda|x-y|^2}{4(s-t)}},\\ |\Gamma_x^0(t,x;s,y)|\le\dfrac{K}{(s-t)^{\frac{n+2}{2}}}\,e^{-\frac{\lambda|x-y|^2}{4(s-t)}},\end{cases}\qquad \lambda<\lambda_0. \tag{5.23}$$

Moreover,

$$\begin{aligned}\Gamma_y^0(t,x;s,y)&=-\Gamma_x^0(t,x;s,y)-\Gamma^0(t,x;s,y)\Big[\big(\det\big[a(s,y)\big]\big)_y+\langle\,\big[a(s,y)^{-1}\big]_y(x-y),x-y\,\rangle\Big]\\ &\equiv-\Gamma_x^0(t,x;s,y)-\Gamma^0(t,x;s,y)\rho(t,x,s,y),\end{aligned} \tag{5.24}$$

where

$$\langle \big[a(s,y)^{-1}\big]_y (x-y), x-y \rangle = \begin{pmatrix} \langle [a(s,y)^{-1}]_{y_1}(x-y), x-y \rangle \\ \langle [a(s,y)^{-1}]_{y_2}(x-y), x-y \rangle \\ \vdots \\ \langle [a(s,y)^{-1}]_{y_n}(x-y), x-y \rangle \end{pmatrix},$$

and

$$\rho(t,x,s,y) = \frac{1}{2}\big(\det\big[a(s,y)\big]\big)_y + \frac{\langle \big[a(s,y)^{-1}\big]_y (x-y), x-y \rangle}{4(s-t)}.$$

Under (P), we see that

$$|\rho(s,x,y)| \le K\Big(1 + \frac{|x-y|^2}{s-t}\Big), \qquad \forall (s,x,y) \in [0,T]\times \mathbb{R}^n \times \mathbb{R}^n. \tag{5.25}$$

Then

$$\begin{aligned}
\Theta_x(\tau,t,x) &= \int_{\mathbb{R}^n} \Gamma^0_x(t,x;T,y)h(\tau,y)dy + \int_t^T\int_{\mathbb{R}^n} \Gamma^0_x(t,x;s,y)\big[\langle b(s,y,v_x(s,y)), \Theta_x(\tau,s,y)\rangle \\
&\qquad\qquad + g(\tau,s,y,v_x(s,y))\big]dyds \\
&= -\int_{\mathbb{R}^n} \Gamma^0_y(t,x;T,y)h(\tau,y)dy - \int_{\mathbb{R}^n} \Gamma^0(t,x;T,y)\rho(t,x,T,y)h(\tau,y)dy \\
&\qquad + \int_t^T\int_{\mathbb{R}^n} \Gamma^0_x(t,x;s,y)\big[\langle b(s,y,v_x(s,y)), \Theta_x(\tau,s,y)\rangle + g(\tau,s,y,v_x(s,y))\big]dyds \\
&= \int_{\mathbb{R}^n} \Gamma^0(t,x;T,y)h_y(\tau,y)dy - \int_{\mathbb{R}^n} \Gamma^0(t,x;s,y)\rho(t,x,T,y)h(\tau,y)dy \\
&\qquad + \int_t^T\int_{\mathbb{R}^n} \Gamma^0_x(t,x;s,y)\big[\langle b(s,y,v_x(s,y)), \Theta_x(\tau,s,y)\rangle + g(\tau,s,y,v_x(s,y))\big]dyds.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\Theta_x(\tau,t,x)| &\le \int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac{n}{2}}}\Big[|h_y(\tau,y)| + \Big(1+\frac{|x-y|^2}{T-t}\Big)|h(\tau,y)|\Big]dy \\
&\qquad + \int_t^T\int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n+1}{2}}}\big(|\Theta_x(\tau,s,y)|+1\big)dyds \\
&\le \int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac{n}{2}}}\Big[\Big(1+\frac{|x-y|^2}{T-t}\Big)|h(\tau,y)| + |h_y(\tau,y)|\Big]dy + \int_t^T\int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n+1}{2}}}|\Theta_x(\tau,s,y)|dyds \\
&\le K\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big) + \int_t^T\int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n+1}{2}}}|\Theta_x(\tau,s,y)|dyds.
\end{aligned} \tag{5.26}$$

The above can be iterated as follows:

$$
\begin{aligned}
&|\Theta_x(\tau,t,x)| \le K\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big) + \int_t^T \int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n+1}{2}}} |\Theta_x(\tau,s,y)|dyds \\
&\le K\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big) + \int_t^T \int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n+1}{2}}} K\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big)dyds \\
&\qquad + \int_t^T \int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n+1}{2}}} \int_s^T \int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|y-z|^2}{4(r-s)}}}{(r-s)^{\frac{n+1}{2}}} |\Theta_x(\tau,r,z)|dzdrdyds \\
&\le K\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big) \\
&\qquad + \int_t^T \int_{\mathbb{R}^n} \Big(\int_t^r \int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n+1}{2}}} \frac{Ke^{-\frac{\lambda|y-z|^2}{4(r-s)}}}{(r-s)^{\frac{n+1}{2}}} dyds\Big) |\Theta_x(\tau,r,z)|dzdr \\
&\le K\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big) + \int_t^T \int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-z|^2}{4(r-t)}}}{(r-t)^{\frac{n}{2}}} |\Theta_x(\tau,r,z)|dzdr.
\end{aligned}
\tag{5.27}
$$

In the above, we have used Lemma 3 of Chapter 1 in [9]. We can repeat the above procedure $2n$ times and then use Gronwall's inequality to obtain

$$
|\Theta_x(\tau,t,x)| \le K\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big), \qquad \forall (t,x)\in[\tau,T]\times\mathbb{R}^n. \tag{5.28}
$$

Now, we let $v^i(\cdot\,,\cdot)\in C([0,T];C^1(\mathbb{R}^n))$, $i=0,1$. Let $\Theta^i(\tau,\cdot\,,\cdot)$ be the corresponding solutions of (5.20). Then

$$
\begin{aligned}
\Theta^1(\tau,t,x)-\Theta^0(\tau,t,x) = \int_t^T\!\!\int_{\mathbb{R}^n} &\Gamma^0(t,x;s,y)\big[\,\langle\, b(s,y,v_x^1(s,y)), \Theta_x^1(\tau,s,y)-\Theta_x^0(\tau,s,y)\,\rangle \\
&+\langle\, b(s,y,v_x^1(s,y))-b(s,y,v_x^0(s,y)), \Theta_x^0(\tau,s,y)\,\rangle \\
&+g(\tau,s,y,v_x^1(s,y))-g(\tau,s,y,v^0(s,y))\big]dyds, \quad (t,x)\in[\tau,T]\times\mathbb{R}^n,
\end{aligned}
\tag{5.29}
$$

and

$$
\begin{aligned}
\Theta_x^1(\tau,t,x)-\Theta_x^0(\tau,t,x) = \int_t^T\!\!\int_{\mathbb{R}^n} &\Gamma_x^0(t,x;s,y)\big[\,\langle\, \Theta_x^1(\tau,s,y)-\Theta_x^0(\tau,s,y), b(s,y,v_x^1(s,y))\,\rangle \\
&+\langle\, \Theta_x^0(\tau,s,y), b(s,y,v_x^1(s,y))-b(s,y,v_x^0(s,y))\,\rangle \\
&+g(\tau,s,y,v_x^1(s,y))-g(\tau,s,y,v^0(s,y))\big]dyds, \quad (t,x)\in[\tau,T]\times\mathbb{R}^n,
\end{aligned}
\tag{5.30}
$$

Hence,

$$
\begin{aligned}
|\Theta_x^1(\tau,t,x)-\Theta_x^0(\tau,t,x)| &\le \int_t^T\!\!\int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n+1}{2}}} \Big(|\Theta_x^1(\tau,s,y)-\Theta_x^0(\tau,s,y)| \\
&\qquad +\big(|\Theta_x^0(\tau,s,y)|+1\big)\,|v_x^1(s,y)-v_x^0(s,y)|\Big)dyds \\
&\le \int_t^T\!\!\int_{\mathbb{R}^n} \frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n+1}{2}}} |\Theta_x^1(\tau,s,y)-\Theta_x^0(\tau,s,y)|dyds \\
&\qquad +K(T-t)^{\frac12}\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big)\|v_x^1(\cdot\,,\cdot)-v_x^0(\cdot\,,\cdot)\|_{C([\tau,T]\times\mathbb{R}^n)}.
\end{aligned}
\tag{5.31}
$$

Then, similar to (5.28) obtained from (5.26) via (5.27), we can obtain

$$
|\Theta_x^1(\tau,t,x)-\Theta_x^0(\tau,t,x)| \le K(T-t)^{\frac12}\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big)\|v_x^1(\cdot\,,\cdot)-v_x^0(\cdot\,,\cdot)\|_{C([\tau,T]\times\mathbb{R}^n)}. \tag{5.32}
$$

On the other hand, from (5.29), we have

$$|\Theta^1(\tau,t,x)-\Theta^0(\tau,t,x)|\le\int_t^T\!\!\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(s-t)}}}{(s-t)^{\frac{n}{2}}}\Big(|\Theta^1_x(\tau,s,y)-\Theta^0_x(\tau,s,y)| \\ +\big(|\Theta^0_x(\tau,s,y)|+1\big)|v^1_x(s,y)-v^0_x(s,y)|\Big)dyds. \tag{5.33}$$

Comparing the above with (5.31), we see that the following must be true:

$$|\Theta^1(\tau,t,x)-\Theta^0(\tau,t,x)|\le K(T-t)\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big)\|v^1_x(\cdot\,,\cdot)-v^0_x(\cdot\,,\cdot)\|_{C([\tau,T]\times\mathbb{R}^n)}. \tag{5.34}$$

Hence, we obtain

$$\begin{aligned}&\|\Theta^1(\tau,\cdot\,,\cdot)-\Theta^0(\tau,\cdot\,,\cdot)\|_{C([\tau,T];C^1(\mathbb{R}^n))}\\ &\le K(T-t)^{\frac12}\big(1+\|h(\tau,\cdot)\|_{C^1(\mathbb{R}^n)}\big)\|v^1(\cdot\,,\cdot)-v^0(\cdot\,,\cdot)\|_{C([\tau,T];C^1(\mathbb{R}^n))}.\end{aligned} \tag{5.35}$$

From the above procedure, we see that $K>0$ in the above is an absolute constant, independent of $(\tau,t)\in D[0,T]$. Hence, in particular, we have (denoting $V^i(t,x)=\Theta^i(t,t,x)$)

$$\begin{aligned}&\|V^1(\cdot\,,\cdot)-V^0(\cdot\,,\cdot)\|_{C([T-\delta,T];C^1(\mathbb{R}^n))}\\ &\le K\delta^{\frac12}\big(1+\|h(\cdot\,,\cdot)\|_{B([0,T];C^1(\mathbb{R}^n))}\big)\|v^1(\cdot\,,\cdot)-v^0(\cdot\,,\cdot)\|_{C([T-\delta,T];C^1(\mathbb{R}^n))}.\end{aligned} \tag{5.36}$$

Clearly, by choosing $\delta>0$ small, we get a contraction mapping $v(\cdot\,,\cdot)\mapsto V(\cdot\,,\cdot)$ on $C([T-\delta,T];C^1(\mathbb{R}^n))$. Therefore, this map admits a unique fixed point. Since we may obtain similar estimates on $[T-2\delta,T-\delta])$, etc., one sees that the fixed point will exists on the whole space $C([0,T;C^1(\mathbb{R}^n))$ for the map $v(\cdot\,,\cdot)\mapsto V(\cdot\,,\cdot)$. Then we obtain the well-posedness of the following:

$$\Theta(\tau,t,x)=\int_{\mathbb{R}^n}\Gamma^0(t,x;T,y)h(\tau,y)dy+\int_t^T\!\!\int_{\mathbb{R}^n}\Gamma^0(t,x;s,y)\big[\,\langle\, b(s,y,\Theta_x(s,s,y)),\Theta_x(\tau,s,y)\,\rangle \\ +g(\tau,s,y,\Theta_x(s,s,y))\big]dyds,\qquad(\tau,t,x)\in D[0,T]\times\mathbb{R}^n, \tag{5.37}$$

Finally, by the regularity of the above expression, we know that $\Theta(\tau,t,x)$ is $C^{2+\alpha}$ in x, $C^{1+\frac{\alpha}{2}}$ in t for some $\alpha\in(0,1)$, and the PDE (5.8) is satisfied. □

Recall that in Section 4, by assuming the convergence of $\Theta^\Pi(\cdot\,,\cdot\,,\cdot)$ and $V^\Pi(\cdot\,,\cdot)$, we get the equilibrium HJB equation for $\Theta(\cdot\,,\cdot\,,\cdot)$ and then $V(\cdot\,,\cdot)$ is characterized by an equilibrium HJB integral equation. We now want to show that under conditions ensuring (P), we do have the expected convergence. This will make our whole procedure satisfactorily complete for certain cases, at least. For the sake of simplicity, we assume that all the involved functions are bounded and continuously differentiable up to a needed order with bounded derivatives.

When (5.6) holds, for $k=0,1,\cdots,N-1$, we have

$$\begin{cases}\Theta^\Pi_t(\tau,t,x)+\mathrm{tr}\,\big[a(t,x)\Theta^\Pi_{xx}(\tau,t,x)\big]+\langle\, b\big(t,x,\psi(\ell^\Pi(t),t,x,V^\Pi_x(t,x))\big),\Theta^\Pi_x(\tau,t,x)\,\rangle\\ \qquad\qquad+g^\Pi\big(\tau,t,x,\psi(\ell^\Pi(t),t,x,V^\Pi_x(t,x))\big)=0,\qquad(t,x)\in[t_k,T]\times\mathbb{R}^n,\\ \Theta^\Pi(\tau,T,x)=h^\Pi(\tau,x),\qquad\qquad x\in\mathbb{R}^n,\end{cases} \tag{5.38}$$

with

$$\begin{cases}h^\Pi(\tau,x)=\displaystyle\sum_{k=1}^{N-1}h(t_{k-1},x)I_{[t_{k-1},t_k)}(\tau),\qquad(\tau,x)\in[0,T]\times\mathbb{R}^n,\\ g^\Pi(\tau,s,x,u)=\displaystyle\sum_{k=1}^{N-1}g(t_{k-1},s,x,u)I_{[t_{k-1},t_k)}(\tau),\qquad(\tau,s,x,u)\in D[0,T]\times\mathbb{R}^n\times U.\end{cases}$$

Since (5.38) is well-posed, we have

$$V^{\Pi}(t,x)=\Theta^{\Pi}(\ell^{\Pi}(t),t,x),\qquad (t,x)\in[0,t_{N-1})\times\mathbb{R}^n. \tag{5.39}$$

Also, by the assumed uniform Lipschitz continuity of $\tau\mapsto(h(\tau,x),h_y(\tau,y),g(\tau,t,x,u))$, we have

$$\begin{aligned}&|h^{\Pi}(\tau,x)-h(\tau,x)|+|h^{\Pi}_x(\tau,x)-h_x(\tau,x)|+|g^{\Pi}(\tau,s,x,u)-g(\tau,s,x,u)|\le K\|\Pi\|,\\ &\qquad\qquad\qquad\forall(\tau,s,x,u)\in D[0,T]\times\mathbb{R}^n\times U.\end{aligned} \tag{5.40}$$

Next, by Proposition 5.1, we have

$$\begin{aligned}&\Theta^{\Pi}(\tau,t,x)-\Theta(\tau,t,x)=\int_{\mathbb{R}^n}\Gamma^0(t,x;T,y)\big[h^{\Pi}(\tau,y)-h(\tau,y)\big]dy\\
&\quad+\int_t^T\!\!\int_{\mathbb{R}^n}\Gamma^0(t,x;s,y)\big[\,\langle\, b\big(s,y,\psi(\ell^{\Pi}(s),s,y,V^{\Pi}_x(s,y))\big),\Theta^{\Pi}_x(\tau,s,y)\,\rangle\\
&\qquad\qquad-\langle\, b\big(s,y,\psi(s,s,y,V_x(s,y))\big),\Theta_x(\tau,s,y)\,\rangle\\
&\qquad\qquad+g^{\Pi}\big(\tau,s,y,\psi(\ell^{\Pi}(s),x,V^{\Pi}_x(s,y))\big)-g\big(\tau,s,y,\psi(s,x,V_x(s,y))\big)\big]dyds\\
&=\int_{\mathbb{R}^n}\Gamma^0(t,x;T,y)\big[h^{\Pi}(\tau,y)-h(\tau,y)\big]dy\\
&\quad+\int_t^T\!\!\int_{\mathbb{R}^n}\Gamma^0(t,x;s,y)\big[\,\langle\, b\big(s,y,\psi(\ell^{\Pi}(s),s,y,V^{\Pi}_x(s,y))\big),\Theta^{\Pi}_x(\tau,s,y)-\Theta_x(\tau,s,y)\,\rangle\\
&\qquad\qquad+\langle\, b\big(s,y,\psi(\ell^{\Pi}(s),s,y,V^{\Pi}_x(s,y))\big)-b\big(s,y,\psi(s,s,y,V_x(s,y))\big),\Theta_x(\tau,s,y)\,\rangle\\
&\qquad\qquad+g^{\Pi}\big(\tau,s,y,\psi(\ell^{\Pi}(s),x,V^{\Pi}_x(s,y))\big)-g\big(\tau,s,y,\psi(s,x,V_x(s,y))\big)\big]dyds.\end{aligned} \tag{5.41}$$

Consequently,

$$\begin{aligned}&\Theta^{\Pi}_x(\tau,t,x)-\Theta_x(\tau,t,x)=\int_{\mathbb{R}^n}\Gamma^0(t,x;T,y)\big[h^{\Pi}_y(\tau,y)-h_y(\tau,y)\big]dy\\
&\quad-\int_{\mathbb{R}^n}\Gamma^0(t,x;T,y)\rho(t,x,T,y)\big[h^{\Pi}(\tau,y)-h(\tau,y)\big]dy\\
&\quad+\int_t^T\!\!\int_{\mathbb{R}^n}\Gamma^0_x(t,x;s,y)\big[\,\langle\, b\big(s,y,\psi(\ell^{\Pi}(s),s,y,V^{\Pi}_x(s,y))\big),\Theta^{\Pi}_x(\tau,s,y)-\Theta_x(\tau,s,y)\,\rangle\\
&\qquad\qquad+\langle\, b\big(s,y,\psi(\ell^{\Pi}(s),s,y,V^{\Pi}_x(s,y))\big)-b\big(s,y,\psi(s,s,y,V_x(s,y))\big),\Theta_x(\tau,s,y)\,\rangle\\
&\qquad\qquad+g^{\Pi}\big(\tau,s,y,\psi(\ell^{\Pi}(s),s,x,V^{\Pi}_x(s,y))\big)-g\big(\tau,s,y,\psi(s,x,V_x(s,y))\big)\big]dyds.\end{aligned} \tag{5.42}$$

Thus,

$$\begin{aligned}&|\Theta^{\Pi}_x(\tau,t,x)-\Theta_x(\tau,t,x)|\le\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac n2}}|h^{\Pi}_y(\tau,y)-h_y(\tau,y)|dy\\
&\quad+\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac n2}}\Big(1+\frac{|x-y|^2}{(T-t)}\Big)|h^{\Pi}(\tau,y)-h(\tau,y)|dy\\
&\quad+\int_t^T\!\!\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac{n+1}{2}}}\Big(|\Theta^{\Pi}_x(\tau,s,y)-\Theta_x(\tau,s,y)|+|\ell^{\Pi}(s)-s|+|V^{\Pi}_x(s,y)-V_x(s,y)|\\
&\qquad\qquad+|g^{\Pi}\big(\tau,s,y,\psi(\tau,s,x,V_x(s,y))\big)-g\big(\tau,s,y,\psi(s,x,V_x(s,y))\big)|\Big)dyds\\
&\le\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac n2}}\|\Pi\|dy+\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac n2}}\Big(1+\frac{|x-y|^2}{(T-t)}\Big)\|\Pi\|dy\\
&\quad+\int_t^T\!\!\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac{n+1}{2}}}\Big(|\Theta^{\Pi}_x(\tau,s,y)-\Theta_x(\tau,s,y)|+\|\Pi\|+|\Theta^{\Pi}_x(\ell^{\Pi}(s),s,y)-\Theta_x(s,s,y)|\Big)dyds.\end{aligned} \tag{5.43}$$

Therefore,

$$\begin{aligned}\sup_{\tau\in[0,t]}|\Theta_x^\Pi(\tau,t,x)-\Theta_x(\tau,t,x)|\le K\|\Pi\|+&\int_t^T\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac{n+1}{2}}}|\Theta_x(\ell^\Pi(s),s,x)-\Theta_x(s,s,y)|dyds\\&+\int_t^T\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac{n+1}{2}}}\sup_{\tau\in[0,s]}|\Theta_x^\Pi(\tau,s,y)-\Theta_x(\tau,s,y)|dyds.\end{aligned}\tag{5.44}$$

This yields that

$$\sup_{\tau\in[0,t]}|\Theta_x^\Pi(\tau,t,x)-\Theta_x(\tau,t,x)|\le K\Big(\|\Pi\|+\int_t^T\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac{n+1}{2}}}|\Theta_x(\ell^\Pi(s),s,x)-\Theta_x(s,s,y)|dyds\Big).\tag{5.45}$$

Likewise, we can have a similar estimate for $|\Theta^\Pi(\tau,t,x)-\Theta(\tau,t,x)|$. Hence, we obtain

$$\begin{aligned}&\sup_{(\tau,t)\in D[0,T]}\|\Theta^\Pi(\tau,t,\cdot)-\Theta(\tau,t,\cdot)\|_{C^1(\mathbb{R}^n)}\\&\le K\Big(\|\Pi\|+\int_t^T\int_{\mathbb{R}^n}\frac{Ke^{-\frac{\lambda|x-y|^2}{4(T-t)}}}{(T-t)^{\frac{n+1}{2}}}|\Theta_x(\ell^\Pi(s),s,x)-\Theta_x(s,s,y)|dyds\Big).\end{aligned}\tag{5.46}$$

From this, our expected convergence follows.

6 Some Special Cases

In this section, we are going to look at several important special cases. We will mainly look at the corresponding forms of our equilibrium HJB equations.

6.1 A Linear-Quadratic Problem

Let us look at the LQ problem. For any initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$, the state equation is

$$\left\{\begin{aligned}&dX(s)=\big[A(s)X(s)+B(s)u(s)\big]ds+\big[A_1(s)X(s)+B_1(s)u(s)\big]dW(s),\qquad s\in[t,T],\\&X(t)=x,\end{aligned}\right.\tag{6.1}$$

with the cost functional

$$J(t,x;u(\cdot))=\frac12\mathbb{E}_t\Big[\int_t^T\big(\langle Q(t,s)X(s),X(s)\rangle+\langle R(t,s)u(s),u(s)\rangle\big)ds+\langle G(t)X(T),X(T)\rangle\Big].\tag{6.2}$$

Then

$$\begin{aligned}\mathbb{H}(\tau,t,x,u,p,P)&=\langle p,A(t)x+B(t)u\rangle+\frac12\mathrm{tr}\Big[\big(A_1(t)x+B_1(t)u\big)\big(A_1(t)x+B_1(t)u\big)^TP\Big]\\&\qquad+\frac12\Big[\langle Q(\tau,t)x,x\rangle+\langle R(\tau,t)u,u\rangle\Big]\\&=\langle p,A(t)x\rangle+\frac12\langle\big[A_1(t)^TPA_1(t)+Q(\tau,t)\big]x,x\rangle\\&\qquad+\frac12\langle\big[R(\tau,t)+B_1(t)^TPB_1(t)\big]u,u\rangle+\langle u,B(t)^Tp+B_1(t)^TPA_1(t)x\rangle.\end{aligned}$$

This yields

$$\psi(\tau,t,x,p,P)=-\big[R(\tau,t)+B_1(t)^TPB_1(t)\big]^{-1}\big[B(t)^Tp+B_1(t)^TPA_1(t)x\big],$$

and

$$
\begin{aligned}
&\mathbb{H}(\tau,t,x,\psi(t,t,x,\bar p,\bar P),p,P)\\
&=\langle\, p,A(t)x\,\rangle+\frac{1}{2}\,\langle\,\big[A_1(t)^TPA_1(t)+Q(\tau,t)\big]x,x\,\rangle\\
&\qquad+\frac{1}{2}\,\langle\,\big[R(\tau,t)+B_1(t)^TPB_1(t)\big]\psi(t,t,x,\bar p,\bar P),\psi(t,t,x,\bar p,\bar P)\,\rangle\\
&\qquad+\langle\,\psi(t,t,x,\bar p,\bar P),B(t)^Tp+B_1(t)^TPA_1(t)x\,\rangle\\
&=\langle\, p,A(t)x\,\rangle+\frac{1}{2}\,\langle\,\big[A_1(t)^TPA_1(t)+Q(\tau,t)\big]x,x\,\rangle\\
&\quad+\frac{1}{2}\,\langle\,\big[R(\tau,t)+B_1(t)^TPB_1(t)\big]\big[R(t,t)+B_1(t)^T\bar PB_1(t)\big]^{-1}\big[B(t)^T\bar p+B_1(t)^T\bar PA_1(t)x\big],\\
&\qquad\qquad\big[R(t,t)+B_1(t)^T\bar PB_1(t)\big]^{-1}\big[B(t)^T\bar p+B_1(t)^T\bar PA_1(t)x\big]\,\rangle\\
&\quad-\langle\,\big[R(t,t)+B_1(t)^T\bar PB_1(t)\big]^{-1}\big[B(t)^T\bar p+B_1(t)^T\bar PA_1(t)x\big],B(t)^Tp+B_1(t)^TPA_1(t)x\,\rangle\,.
\end{aligned}
$$

Hence, the equilibrium HJB equation takes the following form:

$$
\left\{\begin{aligned}
&\Theta_t(\tau,t,x)+\langle\,\Theta_x(\tau,t,x),A(t)x\,\rangle+\frac{1}{2}\,\langle\,\big[A_1(t)^T\Theta_{xx}(\tau,t,x)A_1(t)+Q(\tau,t)\big]x,x\,\rangle\\
&+\frac{1}{2}\,\langle\,\big[R(\tau,t)+B_1(t)^T\Theta_{xx}(\tau,t,x)B_1(t)\big]\big[R(t,t)+B_1(t)^T\Theta_{xx}(t,t,x)B_1(t)\big]^{-1}\\
&\qquad\cdot\big[B(t)^T\Theta_x(t,t,x)+B_1(t)^T\Theta_{xx}(t,t,x)A_1(t)x\big],\\
&\qquad\big[R(t,t)+B_1(t)^T\Theta_{xx}(t,t,x)B_1(t)\big]^{-1}\big[B(t)^T\Theta_x(t,t,x)+B_1(t)^T\Theta_{xx}(t,t,x)A_1(t)x\big]\,\rangle\\
&-\langle\,\big[R(t,t)+B_1(t)^T\Theta_{xx}(t,t,x)B_1(t)\big]^{-1}\big[B(t)^T\Theta_x(t,t,x)+B_1(t)^T\Theta_{xx}(t,t,x)A_1(t)x\big],\\
&\qquad B(t)^T\Theta_x(\tau,t,x)+B_1(t)^T\Theta_{xx}(\tau,t,x)A_1(t)x\,\rangle=0,\qquad(\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\
&\Theta(\tau,T,x)=\frac{1}{2}\,\langle\,G(\tau)x,x\,\rangle,\qquad(\tau,x)\in[0,T]\times\mathbb{R}^n.
\end{aligned}\right.\tag{6.3}
$$

Although having a little bit complicated looking, the above has a quadratic structure which can help us to study the well-posedness of it. To see that, let

$$
\Theta(\tau,t,x)=\frac{1}{2}\,\langle\,P(\tau,t)x,x\,\rangle,\qquad(\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\tag{6.4}
$$

with some undetermined map $P:D[0,T]\to\mathcal{S}^n$. Plugging the above into (6.3), we see that the map $P(\cdot\,,\cdot)$ should satisfy the following equation:

$$
\begin{aligned}
0&=P_t(\tau,t)+P(\tau,t)A(t)+A(t)^TP(\tau,t)+A_1(t)^TP(\tau,t)A_1(t)+Q(\tau,t)\\
&\quad+[P(t,t)B(t)+A_1(t)^TP(t,t)B_1(t)][R(t,t)+B_1(t)^TP(t,t)B_1(t)]^{-1}\\
&\quad\cdot[R(\tau,t)+B_1(t)^TP(\tau,t)B_1(t)][R(t,t)+B_1(t)^TP(t,t)B_1(t)]^{-1}[B(t)^TP(t,t)+B_1(t)^TP(t,t)A_1(t)]\\
&\quad-[P(t,t)B(t)+A_1(t)^TP(t,t)B_1(t)][R(t,t)+B_1(t)^TP(t,t)B_1(t)]^{-1}[B(t)^TP(\tau,t)+B_1(t)^TP(\tau,t)A_1(t)]\\
&\quad-[P(\tau,t)B(t)+A_1(t)^TP(\tau,t)B_1(t)][R(t,t)+B_1(t)^TP(t,t)B_1(t)]^{-1}[B(t)^TP(t,t)+B_1(t)^TP(t,t)A_1(t)],
\end{aligned}
$$

with the terminal condition

$$
P(\tau,T)=G(\tau).
$$

Note that if $P(t,t)$ and $R(t,t)$ are replaced by $P(\tau,t)$ and $R(\tau,t)$, respectively, the above becomes a standard Riccati equation with a parameter τ. The appearance of $P(t,t)$ and $R(t,t)$ makes the above non-standard.

We may rewrite the above as follows (suppressing t in $P(\tau,t)$, etc., for simplicity):

$$\begin{aligned}
0=&P_t(\tau)+P(\tau)A+A^TP(\tau)+A_1^TP(\tau)A_1+Q(\tau)+\big[P(t)B+A_1^TP(t)B_1\big]\big[R(t)+B_1^TP(t)B_1\big]^{-1}\\
&\cdot\big[R(\tau)+B_1^TP(\tau)B_1\big]\big[R(t)+B_1^TP(t)B_1\big]^{-1}\big[B^TP(t)+B_1^TP(t)A_1\big]\\
&-\big[P(t)B+A_1^TP(t)B_1\big]\big[R(t)+B_1^TP(t)B_1\big]^{-1}\big[B^TP(\tau)+B_1^TP(\tau)A_1\big]\\
&-\big[P(\tau)B+A_1^TP(\tau)B_1\big]\big[R(t)+B_1^TP(t)B_1\big]^{-1}\big[B^TP(t)+B_1^TP(t)A_1\big]\\
=&P_t(\tau)+P(\tau)\Big(A-B\big[R(t)+B_1^TP(t)B_1\big]^{-1}\big[B^TP(t)+B_1^TP(t)A_1\big]\Big)\\
&+\Big(A-B\big[R(t)+B_1^TP(t)B_1\big]^{-1}\big[B^TP(t)+B_1^TP(t)A_1\big]\Big)^TP(\tau)+Q(\tau)\\
&+\Big(A_1-B_1\big[R(t)+B_1^TP(t)B_1\big]^{-1}\big[B^TP(t)+B_1^TP(t)A_1\big]\Big)^TP(\tau)\\
&\cdot\Big(A_1-B_1\big[R(t)+B_1^TP(t)B_1\big]^{-1}\big[B^TP(t)+B_1^TP(t)A_1\big]\Big)\\
&+\big[P(t)B+A_1^TP(t)B_1\big]\big[R(t)+B_1^TP(t)B_1\big]^{-1}R(\tau)\big[R(t)+B_1^TP(t)B_1\big]^{-1}\big[B^TP(t)+B_1^TP(t)A_1\big].
\end{aligned}$$

Denote

$$\begin{cases}\Gamma(t)=\big[R(t,t)+B_1(t)^TP(t,t)B_1(t)\big]^{-1}\big[B(t)^TP(t,t)+B_1(t)^TP(t,t)A_1(t)\big],\\ \widehat A(t)=A(t)-B(t)\Gamma(t),\qquad \widehat A_1(t)=A_1(t)-B_1(t)\Gamma(t),\\ \widehat Q(\tau,t)=Q(\tau,t)+\Gamma(t)^TR(\tau,t)\Gamma(t),\end{cases}\qquad (\tau,t)\in D[0,T].\tag{6.5}$$

Then the equation for $P(\cdot\,,\cdot)$ can be written as follows:

$$\begin{cases}P_t(\tau,t)+P(\tau,t)\widehat A(t)+\widehat A(t)^TP(\tau,t)+\widehat A_1(t)^TP(\tau,t)\widehat A_1(t)+\widehat Q(\tau,t)=0,\qquad (\tau,t)\in D[0,T],\\ P(\tau,T)=G(\tau),\qquad \tau\in[0,T].\end{cases}\tag{6.6}$$

Next, let $\Phi(\cdot\,,\cdot)$ be the fundamental matrix of $\big(\widehat A(\cdot),\widehat A_1(\cdot)\big)$, i.e., the following holds:

$$\begin{cases}d\Phi(s,t)=\widehat A(s)\Phi(s,t)ds+\widehat A_1(s)\Phi(s,t)dW(s),\qquad s\in[t,T],\\ \Phi(t,t)=I.\end{cases}\tag{6.7}$$

Applying Itô's formula to $s\mapsto\langle\,P(\tau,s)\Phi(s,t)x,\Phi(s,t)x\,\rangle$ on $[t,T]$, we have

$$\begin{aligned}\langle\,\Phi(T,t)^TG(\tau)\Phi(T,t)x,x\,\rangle-\langle\,P(\tau,t)x,x\,\rangle=&\int_t^T-\langle\,\widehat Q(\tau,s)\Phi(s,t)x,\Phi(s,t)x\,\rangle\,ds\\ &+\int_t^T\langle\,\big[P(\tau,s)\widehat A_1(s)+\widehat A_1(s)^TP(\tau,s)\big]\Phi(s,t)x,\Phi(s,t)x\,\rangle\,dW(s),\end{aligned}$$

which leads to

$$\begin{aligned}P(\tau,t)=&\Phi(T,t)^TG(\tau)\Phi(T,t)+\int_t^T\Phi(s,t)^T\widehat Q(\tau,s)\Phi(s,t)ds\\ &-\int_t^T\Phi(s,t)^T\big[P(\tau,s)\widehat A_1(s)+\widehat A_1(s)^TP(\tau,s)\big]\Phi(s,t)dW(s),\qquad (\tau,t)\in D[0,T].\end{aligned}\tag{6.8}$$

Note that although $P(\tau,t)$ is a deterministic function, the above representation is stochastic. From the above, we have

$$P(\tau,t)=\mathbb{E}_t\Big[\Phi(T,t)^TG(\tau)\Phi(T,t)+\int_t^T\Phi(s,t)^T\big[Q(\tau,s)+\Gamma(s)^TR(\tau,s)\Gamma(s)\big]\Phi(s,t)ds\Big],\qquad (\tau,t)\in D[0,T].\tag{6.9}$$

In particular, taking $\tau = t$ and denoting $P(t) = P(t,t)$, one has

$$P(t) = \mathbb{E}_t\Big[\Phi(T,t)^T G(t)\Phi(T,t) + \int_t^T \Phi(s,t)^T \big[Q(t,s) + \Gamma(s)^T R(t,s)\Gamma(s)\big]\Phi(s,t)ds\Big], \qquad t \in [0,T]. \tag{6.10}$$

Combining the above, we end up with the following system for the function $P(\cdot)$:

$$\begin{cases} P(t) = \mathbb{E}_t\Big[\Phi(T,t)^T G(t)\Phi(T,t) + \displaystyle\int_t^T \Phi(s,t)^T \big[Q(t,s) + \Gamma(s)^T R(t,s)\Gamma(s)\big]\Phi(s,t)ds\Big], \qquad t \in [0,T], \\ \Phi(s,t) = I + \displaystyle\int_t^s \big[A(r) - B(r)\Gamma(r)\big]\Phi(r,t)dr + \int_t^s \big[A_1(r) - B_1(r)\Gamma(r)\big]\Phi(r,t)dW(r), \quad (t,s) \in D[0,T], \\ \Gamma(t) = \big[R(t,t) + B_1(t)^T P(t)B_1(t)\big]^{-1}\big[B(t)^T P(t) + B_1(t)^T P(t)A_1(t)\big], \qquad t \in [0,T]. \end{cases} \tag{6.11}$$

We refer to the above as a *Riccati-Volterra integral equation system* for the corresponding (time-inconsistent) LQ problem. Note that the above is actually a coupled forward-backward stochastic Volterra integral equation system (FBSVIE, for short). Some relevant results concerning backward stochastic Volterra integral equations (BSVIEs, for short) can be found in [25, 26]. If $\big(\Phi(\cdot\,,\cdot), P(\cdot)\big)$ is a solution to the above, then the time-consistent equilibrium control is given by

$$\bar u(t) = -\Gamma(t)\bar X(t), \quad t \in [0,T]. \tag{6.12}$$

In the case that

$$A_1(\cdot) = 0, \qquad B_1(\cdot) = 0, \tag{6.13}$$

the above (6.11) is reduced to the following Riccati-Volterra integral equation system (for a deterministic time-inconsistent LQ problem):

$$\begin{cases} P(t) = \Phi(T,t)^T G(t)\Phi(T,t) + \displaystyle\int_t^T \Phi(s,t)^T \big[Q(t,s) + \Gamma(s)^T R(t,s)\Gamma(s)\big]\Phi(s,t)ds, \quad t \in [0,T], \\ \Phi(s,t) = I + \displaystyle\int_t^s \big[A(r) - B(r)\Gamma(r)\big]\Phi(r,t)dr, \qquad (t,s) \in D[0,T], \\ \Gamma(t) = R(t,t)^{-1}B(t)^T P(t), \qquad t \in [0,T], \end{cases} \tag{6.14}$$

and the time-consistent equilibrium control is given by (6.12) with a simpler $\Gamma(\cdot)$. This recovers the case presented in [27] where the well-posedness of (6.14) was established.

For (6.11), we have the following result.

Proposition 6.1. *Suppose*

$$\begin{cases} A(\cdot), A_1(\cdot) \in C([0,T];\mathbb{R}^{n\times n}), \quad B(\cdot), B_1(\cdot) \in C([0,T];\mathbb{R}^{n\times m}), \\ Q(\cdot) \in C(D[0,T];\overline{\mathcal{S}^n_+}), \quad R(\cdot) \in C(D[0,T];\mathcal{S}^m_+), \quad G(\cdot) \in C([0,T];\overline{\mathcal{S}^n_+}). \end{cases} \tag{6.15}$$

Further, suppose

$$\sup_{P\in\mathcal{S}^n_+}\Big|\big[R(t,t) + B_1(t)^T P B_1(t)\big]^{-1}\big[B(t)^T P + B_1(t)^T P A_1(t)\big]\Big| \equiv L < \infty. \tag{6.16}$$

Then (6.11) *admits a unique solution.*

Proof. Let $\mathcal{X}[\tau,T] \equiv C\big([\tau,T];\overline{\mathcal{S}^n_+}\big)$ which is a complete metric space with the metric induced by the norm in $C([\tau,T];\mathcal{S}^n)$. For any $p(\cdot) \in \mathcal{X}[0,T]$, define

$$\Gamma(t;p(\cdot)) = \big[R(t,t) + B_1(t)^T p(t)B_1(t)\big]^{-1}\big[B(t)^T p(t) + B_1(t)^T p(t)A_1(t)\big], \qquad t \in [0,T].$$

By (6.16), we have

$$|\Gamma(t;p(\cdot))| \le K, \qquad t\in[0,T],$$

with the bound independent of $p(\cdot)\in\mathcal{X}$. Let $\Phi(\cdot\,,\cdot)\equiv\Phi(\cdot\,,\cdot\,;p(\cdot))$ be the solution to the following:

$$\begin{aligned}\Phi(s,t)=I+&\int_t^s\big[A(r)-B(r)\Gamma(r;p(\cdot))\big]\Phi(r,t)dr\\&+\int_t^s\big[A_1(r)-B_1(r)\Gamma(r;p(\cdot))\big]\Phi(r,t)dW(r),\qquad (t,s)\in D[0,T].\end{aligned}$$

We have

$$\mathbb{E}\Big[\sup_{(t,s)\in D[0,T]}|\Phi(s,t)|^2\Big]\le K,$$

for some absolute constant K. Next, we define

$$\begin{aligned}P(t)\equiv P(t;p(\cdot))=\mathbb{E}_t\Big[&\Phi(T,t)^TG(t)\Phi(T,t)\\&+\int_t^T\Phi(s,t)^T\big[Q(t,s)+\Gamma(s;p(\cdot))^TR(t,s)\Gamma(s;p(\cdot))\big]\Phi(s,t)ds,\qquad t\in[0,T].\end{aligned}$$

Clearly, $P(\cdot)\in\mathcal{X}[0,T]$. We want to show that $p(\cdot)\mapsto P(\cdot\,;p(\cdot))$ admits a unique fixed point. To this end, for any $p^1(\cdot),p^2(\cdot)\in\mathcal{X}[0,T]$, we denote

$$\Gamma^i(\cdot)=\Gamma(\cdot\,;p^i(\cdot)),\quad \Phi^i(\cdot\,,\cdot)=\Phi(\cdot\,,\cdot\,;p^i(\cdot)),\quad P^i(\cdot)=P(\cdot\,;p^i(\cdot)),\qquad i=1,2.$$

Then (suppressing t)

$$\begin{aligned}&|\Gamma^1(t)-\Gamma^2(t)|=\big|(R+B_1^Tp_1B_1)^{-1}(B^Tp_1+B_1^Tp_1A_1)-(R+B_1^Tp_2B_1)^{-1}(B^Tp_2+B_1^Tp_2A_1)|\\&\le\big|(R+B_1^Tp_1B_1)^{-1}\big|\,\big|B^T(p_1-p_2)+B_1^T(p_1-p_2)A_1\big|\\&\qquad+\big|\big[(R+B_1^Tp_1B_1)^{-1}-(R+B_1^Tp_2B_1)^{-1}\big](B^Tp_2+B_1^Tp_2B_1)\big|\\&\le K|p_1-p_2|+\big|(R+B_1^Tp_1B_1)^{-1}B_1^T(p_1-p_2)B_1(R+B_1^Tp_2B_1)^{-1}(B^Tp_2+B_1^Tp_2A_1)\big|\\&\le K|p_1(t)-p_2(t)|.\end{aligned}$$

Next,

$$\begin{aligned}\Phi^1(s,t)-\Phi^2(s,t)=&\int_t^s\big\{A(r)\big[\Phi^1(r,t)-\Phi^2(r,t)\big]-B(r)\big[\Gamma^1(r)-\Gamma^2(r)\big]\Phi^1(r,t)\\&\qquad-B(r)\Gamma^2(r)\big[\Phi^1(r,t)-\Phi^2(r,t)\big]\big\}dr\\&+\int_t^s\big\{A_1(r)\big[\Phi^1(r,t)-\Phi^2(r,t)\big]-B_1(r)\big[\Gamma^1(r)-\Gamma^2(r)\big]\Phi^1(r,t)\\&\qquad-B_1(r)\Gamma^2(r)\big[\Phi^1(r,t)-\Phi^2(r,t)\big]\big\}dW(r).\end{aligned}$$

Then

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|\Phi^1(s,t)-\Phi^2(s,t)|^2\Big]\le K\int_t^s|\Gamma^1(r)-\Gamma^2(r)|^2dr\le K\int_t^s|p^1(r)-p^2(r)|^2dr.$$

Consequently,

$$
\begin{aligned}
|P^1(t)-P^2(t)| &\le \mathbb{E}\Big\{\big|\Phi^1(T,t)^TG(t)\Phi^1(T,t)-\Phi^2(T,t)G(t)\Phi^2(T,t)\big| \\
&\qquad\qquad +\int_t^T\big|\Phi^1(s,t)\big[Q(t,s)+\Gamma^1(s)R(t,s)\Gamma^1(s)\big]\Phi^1(r,t) \\
&\qquad\qquad\qquad -\Phi^2(s,t)\big[Q(t,s)+\Gamma^2(s)R(t,s)\Gamma^2(s)\big]\Phi^2(r,t)\big|dr\Big\} \\
&\le K\mathbb{E}\Big\{\Big(|\Phi^1(T,t)|+|\Phi^2(T,t)|\Big)|\Phi^1(T,t)-\Phi^2(T,t)| \\
&\quad +\int_t^T\Big[\Big(|\Phi^1(s,t)|+|\Phi^2(s,t)|\Big)|\Phi^1(s,t)-\Phi^2(s,t)| \\
&\qquad +\Big(|\Phi^1(s,t)\Gamma^1(s)|+|\Phi^2(s,t)\Gamma^2(s)|\Big)|\Phi^1(s,t)\Gamma^1(s)-\Phi^2(s,t)\Gamma^2(s)|\Big]ds\Big\} \\
&\le K\Big\{\Big(\mathbb{E}|\Phi^1(T,t)-\Phi^2(T,t)|^2\Big)^{\frac12}+\int_t^T\Big[\Big(\mathbb{E}|\Phi^1(s,t)-\Phi^2(s,t)|^2\Big)^{\frac12} \\
&\qquad +\Big(\mathbb{E}|\Phi^1(s,t)\Gamma^1(s)-\Phi^2(s,t)\Gamma^2(s)|^2\Big)^{\frac12}\Big]ds\Big\} \\
&\le K\Big\{\int_t^T|p^1(s)-p^2(s)|^2ds+\int_t^T\Big[\Big(\int_t^s|p^1(r)-p^2(r)|^2dr\Big)^{\frac12} \\
&\quad +\Big(\mathbb{E}|\Phi^1(s,t)-\Phi^2(s,t)|^2+|\Gamma^1(s)-\Gamma^2(s)|^2\Big)^{\frac12}\Big]ds\Big\}\le K\Big(\int_t^T|p^1(s)-p^2(s)|^2ds\Big)^{\frac12},
\end{aligned}
$$

with $K>0$ being an absolute constant. Hence, we obtain

$$\max_{t\in[T-\kappa,T]}|P^1(t)-P^2(t)|\le K(T-\tau)^{\frac12}\max_{t\in[T-\tau,T]}|p^1(t)-p^2(t)|.$$

Hence, the map $p(\cdot)\mapsto P(\cdot)$ is contractive on $\mathcal{X}[\tau,T]$ as long as $T-\tau>0$ small. Then a usual argument applies to obtain a unique fixed point of $p(\cdot)\mapsto P(\cdot)$ on $\mathcal{X}[0,T]$. This proves the well-posedness of (6.11).

$\square$

To conclude this section, let us make a remark on the condition (6.16). It is not hard to show that when $m=n$ and $B_1(\cdot)^{-1}$ exists and bounded, then (6.16) holds. Apparently, this is a restrictive condition. We hope that in our future publications, such a condition can be removed.

6.2 A generalized Merton's problem

In this subsection, let us look at the Merton's portfolio problem with general discounting. The results in this subsection is comparable with some of the results in [7] and [18]. Let us recall

$$\begin{cases} dX(s)=\big[rX(s)+(\mu-r)u(s)-c(s)\big]ds+\sigma u(s)dW(s), & s\in[t,T], \\ X(t)=x, \end{cases}\tag{6.17}$$

and

$$J(t,x;u(\cdot),c(\cdot))=\mathbb{E}_t\Big[\int_t^T\nu(t,s)c(s)^\beta ds+\rho(t)X(T)^\beta\Big].\tag{6.18}$$

Then

$$
\begin{aligned}
\mathbb{H}(t,s,x,u,c,p,P)&=p[rx+(\mu-r)u-c]+\frac12\sigma^2u^2P+\nu(t,s)c^\beta \\
&=rxp+\frac{\sigma^2P}{2}\Big[u^2+2\frac{(\mu-r)p}{\sigma^2P}u\Big]+\nu(t,s)c^\beta-pc.
\end{aligned}
$$

The maximum of $(u,c)\mapsto\mathbb{H}(t,s,x,u,c,p,P)$ is attained at

$$\bar u=-\frac{(\mu-r)p}{\sigma^2P},\qquad \bar c=\Big(\frac{\beta\nu(t,s)}{p}\Big)^{\frac{1}{1-\beta}}.$$

We denote

$$\psi(t,s,x,p,P) \equiv (\bar u,\bar c) = \Big(-\frac{(\mu-r)p}{\sigma^2 P}, \Big[\frac{\beta\nu(t,s)}{p}\Big]^{\frac{1}{1-\beta}}\Big).$$

Then

$$\psi(t,t,x,\bar p,\bar P) \equiv (\bar u,\bar c) = \Big(-\frac{(\mu-r)\bar p}{\sigma^2 \bar P}, \Big[\frac{\beta\nu(t,t)}{\bar p}\Big]^{\frac{1}{1-\beta}}\Big).$$

and

$$\begin{aligned}
&\mathbb{H}(\tau,t,x,\psi(t,t,x,\bar p,\bar P),p,P)\\
&= rxp + \frac{\sigma^2 P}{2}\Big[\frac{(\mu-r)^2\bar p^2}{\sigma^4\bar P^2} - 2\frac{(\mu-r)p}{\sigma^2 P}\frac{(\mu-r)\bar p}{\sigma^2\bar P}\Big] + \nu(\tau,t)\Big(\frac{\beta\nu(t,t)}{\bar p}\Big)^{\frac{\beta}{1-\beta}} - p\Big(\frac{\beta\nu(t,t)}{\bar p}\Big)^{\frac{1}{1-\beta}}\\
&= rxp + \frac{(\mu-r)^2\bar p P}{2\sigma^2\bar P}\Big(\frac{\bar p}{\bar P} - 2\frac{p}{P}\Big) + \frac{[\beta\nu(t,t)]^{\frac{\beta}{1-\beta}}}{\bar p^{\frac{1}{1-\beta}}}[\nu(\tau,t)\bar p - \beta\nu(t,t)p].
\end{aligned}$$

Hence, the equilibrium HJB equation reads

$$\begin{cases}
\Theta_t(\tau,t,x) + rx\Theta_x(\tau,t,x) + \dfrac{(\mu-r)^2\Theta_x(t,t,x)\Theta_{xx}(\tau,t,x)}{2\sigma^2\Theta_{xx}(t,t,x)}\Big(\dfrac{\Theta_x(t,t,x)}{\Theta_{xx}(t,t,x)} - 2\dfrac{\Theta_x(\tau,t,x)}{\Theta_{xx}(\tau,t,x)}\Big)\\
+\dfrac{[\beta\nu(t,t)]^{\frac{\beta}{1-\beta}}}{\Theta_x(t,t,x)^{\frac{1}{1-\beta}}}\big[\nu(\tau,t)\Theta_x(t,t,x) - \beta\nu(t,t)\Theta_x(\tau,t,x)\big] = 0, \qquad (\tau,t,x)\in D[0,T]\times(0,\infty),\\
\Theta(\tau,t,0) = 0, \qquad (\tau,t)\in D[0,T],\\
\Theta(\tau,T,x) = \rho(\tau)x^\beta, \qquad (\tau,x)\in(0,\infty)\times[0,\infty).
\end{cases} \tag{6.19}$$

Let

$$\Theta(\tau,t,x) = \varphi(\tau,t)x^\beta, \qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n.$$

Then

$$\begin{aligned}
0 &= \varphi_t(\tau,t) + r\beta\varphi(\tau,t) + \frac{(\mu-r)^2\beta}{2\sigma^2(1-\beta)}\varphi(\tau,t) + \frac{[\beta\nu(t,t)]^{\frac{\beta}{1-\beta}}}{[\beta\varphi(t,t)]^{\frac{1}{1-\beta}}}\big[\nu(\tau,t)\beta\varphi(t,t) - \beta^2\nu(t,t)\varphi(\tau,t)\big]\\
&= \varphi_t(\tau,t) + r\beta\varphi(\tau,t) + \frac{(\mu-r)^2\beta}{2\sigma^2(1-\beta)}\varphi(\tau,t) + \frac{\nu(t,t)^{\frac{\beta}{1-\beta}}}{\varphi(t,t)^{\frac{1}{1-\beta}}}\big[\nu(\tau,t)\varphi(t,t) - \beta\nu(t,t)\varphi(\tau,t)\big]\\
&= \varphi_t(\tau,t) + \beta\Big[r + \frac{(\mu-r)^2}{2\sigma^2(1-\beta)} - \Big(\frac{\nu(t,t)}{\varphi(t,t)}\Big)^{\frac{1}{1-\beta}}\Big]\varphi(\tau,t) + \Big(\frac{\nu(t,t)}{\varphi(t,t)}\Big)^{\frac{\beta}{1-\beta}}\nu(\tau,t)\\
&\equiv \varphi_t(\tau,t) + \Big[\lambda - \beta\Big(\frac{\nu(t,t)}{\varphi(t,t)}\Big)^{\frac{1}{1-\beta}}\Big]\varphi(\tau,t) + \Big(\frac{\nu(t,t)}{\varphi(t,t)}\Big)^{\frac{\beta}{1-\beta}}\nu(\tau,t),
\end{aligned}$$

and

$$\varphi(\tau,T) = \rho(\tau),$$

with λ given by (2.11). Thus,

$$\varphi(\tau,t) = e^{\lambda(T-t)-\beta\int_t^T\left(\frac{\nu(s,s)}{\varphi(s,s)}\right)^{\frac{1}{1-\beta}}ds}\rho(\tau) + \int_t^T e^{\lambda(s-t)-\beta\int_t^s\left(\frac{\nu(s',s')}{\varphi(s',s')}\right)^{\frac{1}{1-\beta}}ds'}\Big(\frac{\nu(s,s)}{\varphi(s,s)}\Big)^{\frac{\beta}{1-\beta}}\nu(\tau,s)ds,$$
$$(\tau,t)\in D[0,T].$$

Hence, we obtain the following integral equation for $t\mapsto\varphi(t,t)$:

$$\begin{aligned}
\varphi(t,t) &= e^{\lambda(T-t)-\beta\int_t^T\left(\frac{\nu(s,s)}{\varphi(s,s)}\right)^{\frac{1}{1-\beta}}ds}\rho(t)\\
&\quad + \int_t^T e^{\lambda(s-t)-\beta\int_t^s\left(\frac{\nu(s',s')}{\varphi(s',s')}\right)^{\frac{1}{1-\beta}}ds'}\Big(\frac{\nu(s,s)}{\varphi(s,s)}\Big)^{\frac{\beta}{1-\beta}}\nu(t,s)ds, \qquad t\in[0,T].
\end{aligned} \tag{6.20}$$

Once such an integral equation admits a unique solution $t \mapsto \varphi(t,t)$, we will obtain the time-consistent equilibrium value function

$$V(t,x) = \Theta(t,t,x) = \varphi(t,t)x^\beta, \qquad (t,x) \in [0,T] \times [0,\infty),$$

and the time-consistent equilibrium control

$$\bar{u}(t) = -\frac{\mu - r}{\sigma^2(1-\beta)}\bar{X}(t), \qquad \bar{c}(t) = \Big(\frac{\nu(t,t)}{\varphi(t,t)}\Big)^{\frac{1}{1-\beta}}\bar{X}(t), \qquad t \in [0,T]. \tag{6.21}$$

To establish the well-posedness of (6.20), we set

$$z(t) = \frac{\varphi(t,t)}{\nu(t,t)}, \qquad t \in [0,T].$$

Then (6.20) is equivalent to the following:

$$z(t) = e^{\lambda(T-t) - \beta\int_t^T z(s)^{\frac{1}{\beta-1}}ds}\rho(t) + \int_t^T e^{\lambda(\tau-t)-\beta\int_t^\tau z(s)^{\frac{1}{\beta-1}}ds} z(\tau)^{\frac{\beta}{\beta-1}}\nu(t,\tau)d\tau, \qquad t \in [0,T]. \tag{6.22}$$

We have the following result.

Proposition 6.2. *Let $\nu : D[0,T] \to (0,\beta]$ and $\rho : [0,T] \to (0,\infty)$ be continuous, and*

$$\bar{\lambda} \equiv \sup_{0 \le t < s \le T} \frac{-\ln \nu(t,s)}{s-t} < \infty. \tag{6.23}$$

Suppose $z : [0,T] \to (0,\infty)$ is a solution to (6.22). Then

$$e^{(\lambda-\bar{\lambda})(T-t)} \min_{t\in[0,T]} \rho(t) \le z(t) \le e^{\lambda(T-t)} \max_{t\in[0,T]} \rho(t), \qquad t \in [0,T], \tag{6.24}$$

Proof. We may write (6.22) as follows:

$$z(t)e^{-\lambda(T-t)+\beta\int_t^T z(s)^{\frac{1}{\beta-1}}ds} = \rho(t) + \int_t^T \nu(t,\tau)z(\tau)^{\frac{1}{\beta-1}}\Big[z(\tau)e^{-\lambda(T-\tau)+\beta\int_\tau^T z(s)^{\frac{1}{\beta-1}}ds}\Big]d\tau.$$

Denoting

$$\widehat{z}(t) = z(t)e^{-\lambda(T-t)+\beta\int_t^T z(s)^{\frac{1}{\beta-1}}ds}, \qquad t \in [0,T],$$

we have

$$\widehat{z}(t) = \rho(t) + \int_t^T \nu(t,\tau)z(\tau)^{\frac{1}{\beta-1}}\widehat{z}(\tau)d\tau, \qquad t \in [0,T]. \tag{6.25}$$

Note that

$$\rho_0 \equiv \min_{t\in[0,T]} \rho(t) \le \rho(t) \le \rho_1 \equiv \max_{t\in[0,T]} \rho(t), \qquad 0 < \nu(t,\tau) \le \beta. \tag{6.26}$$

Thus, (6.25) implies

$$\widehat{z}(t) \le \rho_1 + \beta\int_t^T z(\tau)^{\frac{1}{\beta-1}}\widehat{z}(\tau)d\tau, \qquad t \in [0,T].$$

Then by Gronwall's inequality, we obtain

$$z(t)e^{-\lambda(T-t)+\beta\int_t^T z(s)^{\frac{1}{\beta-1}}ds} \equiv \widehat{z}(t) \le \rho_1 e^{\beta\int_t^T z(s)^{\frac{1}{\beta-1}}ds}, \qquad t \in [0,T],$$

which leads to

$$z(t) \le \rho_1 e^{\lambda(T-t)}, \qquad t \in [0,T].$$

Next, (6.23) implies

$$\nu(t,s) \geq e^{-\bar{\lambda}(s-t)}, \qquad (t,s) \in D[0,T]. \tag{6.27}$$

Consequently, we obtain from (6.25) that

$$\widehat{z}(t) \geq \rho_0 + \int_t^T e^{-\bar{\lambda}(\tau-t)} z(\tau)^{\frac{1}{\beta-1}} \widehat{z}(\tau) d\tau, \qquad t \in [0,T], \tag{6.28}$$

which is equivalent to the following:

$$\widehat{z}(t)e^{-\bar{\lambda}t} \geq \rho_0 e^{-\bar{\lambda}t} + \int_t^T z(\tau)^{\frac{1}{\beta-1}} \left[\widehat{z}(\tau)e^{-\bar{\lambda}\tau}\right] d\tau \equiv \zeta(t).$$

Then

$$\zeta'(t) = -\bar{\lambda}\rho_0 e^{-\bar{\lambda}t} - z(t)^{\frac{1}{\beta-1}} \left[\widehat{z}(t)e^{-\bar{\lambda}t}\right] \leq -\bar{\lambda}\rho_0 e^{-\bar{\lambda}t} - z(t)^{\frac{1}{\beta-1}}\zeta(t),$$

which yields

$$\left[\zeta(t)e^{-\int_t^T z(s)^{\frac{1}{\beta-1}} ds}\right]' \leq -\bar{\lambda}\rho_0 e^{-\bar{\lambda}t - \int_t^T z(s)^{\frac{1}{\beta-1}} ds}.$$

Hence,

$$\rho_0 e^{-\bar{\lambda}T} - \zeta(t)e^{-\int_t^T z(s)^{\frac{1}{\beta-1}} ds} \leq -\bar{\lambda}\rho_0 \int_t^T e^{-\bar{\lambda}\tau} e^{-\int_\tau^T z(s)^{\frac{1}{\beta-1}} ds} d\tau$$

Then

$$\begin{aligned} z(t) &= \widehat{z}(t)e^{\lambda(T-t)} e^{-\int_t^T z(s)^{\frac{1}{\beta-1}} ds} \geq e^{\bar{\lambda}t} e^{\lambda(T-t)} \left[\zeta(t)e^{-\int_t^T z(s)^{\frac{1}{\beta-1}} ds}\right] \\ &\geq e^{\bar{\lambda}t} e^{\lambda(T-t)} \left[\rho_0 e^{-\bar{\lambda}T} + \bar{\lambda}\rho_0 \int_t^T e^{-\bar{\lambda}\tau} e^{-\int_\tau^T z(s)^{\frac{1}{\beta-1}} ds} d\tau\right] \geq \rho_0 e^{(\lambda-\bar{\lambda})(T-t)}, \qquad t \in [0,T]. \end{aligned}$$

This proves our proposition. $\square$

Having the above proposition, we then can easily obtain the well-posedness of (6.20) by means of contraction mapping theorem (giving the local solvability) and a usual continuation argument. Then time-consistent equilibrium control can be constructed by (6.21).

Note that by multiplying a constant to the payoff functional, if necessary, we can always make

$$\max_{(\tau,t) \in D[0,T]} \nu(\tau,t) \leq \beta.$$

On the other hand, in the case that $s \mapsto \nu(s,t)$ is differentiable, we have

$$\lim_{s \downarrow t} \frac{-\ln \nu(t,s)}{s-t} = -\frac{\nu_s(t,t)}{\nu(t,t)}.$$

Thus, condition (6.23) is ensured by the boundedness of the right hand side of the above, which is not too restrictive.

References

[1] S. Basak and G. Chabakauri, *Dynamic mean-variance asset allocation, Rev. Finan. Stud.,* 23 (2010), 2970–3016.

[2] T. Björk and A. Murgoci, *A general theory of Markovian time inconsistent stochasitc control problem,* working paper.

[3] T. Björk, A. Murgoci, and X. Y. Zhou, *Mean varaiance portfolio optimization with state dependent risk aversion, Math. Finance,* in press.

[4] A. Caplin and J. Leahy, *The recursive approach to time inconsistency, J. Econ. Theory*, 131 (2006), 134–156.

[5] I. Ekeland and A. Lazrak, *The golden rule when preferences are time inconsistent, Math. Finan. Econ.*, 4 (2010), 29–55.

[6] I. Ekeland and T. A. Pirvu, *Investment and consumption without commitment, Math. Finan. Econ.*, 2 (2008), 57–86.

[7] I. Ekeland, O. Mbodji, and T. A. Pirvu, *Time-Consistent Portfolio Management, SIAM J. Financial Math.*, 3 (2012), 1–32.

[8] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions, 2nd Edition*, Springer-Verlag, New York, 2006.

[9] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, 1964.

[10] S. M. Goldman, *Consistent plans, Review of Economic Studies*, 47 (1980), 533–537.

[11] S. R. Grenadier and N. Wang, *Investment under uncertainty and time-inconsistent preferences, J. Finan. Econ.*, 84 (2007), 2–39.

[12] P. J. Herings and K. I. M. Rohde, *Time-inconsistent preferences in a general equilibriub model, Econ. Theory*, 29 (2006), 591–619.

[13] Y. Hu, H. Jin, and X. Y. Zhou, *Time-inconsistent stochastic linear-quadratic control*, arXiv:1111.0818v1, 2011.

[14] L. Karp and I. H. Lee, *Time-consistent policies, J. Econ. Theory*, 112 (2003), 353–364.

[15] D. Laibson, *Golden eggs and hyperbolic discounting, Quarterly J. Econ.*, 112 (1997), 443–477.

[16] J. Ma, P. Protter, and J. Yong, *Solving forward-backward stochastic differential equations explicitly — a four-step scheme, Probab. Theory & Related Fields*, 98 (1994), 339–359.

[17] J. Ma and J. Yong, *Forward-Backward Stochastic Differential Equations and Their Applications*, Lecture Notes in Math., Vol. 1702, Springer-Verlag, 1999.

[18] J. Marin-Solano, and J. Navas, *Consumption and portfolio rules for time-inconsistnet investors, European J. Operational Research*, 201 (2010), 860–872.

[19] J. Marin-Solano and E. V. Shevkoplyas, *Non-constant discounting and differntial games with random time horizon, Automaitca*, 47 (2011), 2626–2638.

[20] M. Miller and M. Salmon, *Dynamic games and the time inconsistency of optimal policy in open economics, The Economic Journal*, 95 (1985), 124–137.

[21] I. Palacios-Huerta, *Time-inconsistent preferences in Adam Smith and Davis Hume, History of Political Economy*, 35 (2003), 241–268.

[22] R. A. Pollak, *Consistent planning, Review of Economic Studies*, 35 (1968), 185–199.

[23] R. H. Strotz, *Myopia and inconsistency in dynamic utility maximization, Review of Econ. Studies*, 23 (1955), 165–180.

[24] L. Tesfatsion, *Time inconsistency of benevolent government economics, J. Public Economics*, 31 (1986), 25–52.

[25] J. Yong, *Backward stochastic Volterra integral equations and some related problems, Stoch. Proc. Appl.*, 116 (2006), 779–795.

[26] J. Yong, *Well-posedness and regularity of backward stochastic Volterra integral equations, Prob. Theory Rel. Fields*, 142 (2008), 21–77.

[27] J. Yong, *A Deterministic Linear Quadratic Time-Inconsistent Optimal Control Problem, Math. Control & Related Fields*, 1 (2011), 83–118.

[28] J. Yong, *Deterministic time-inconsistent optimal control problems — An essentially cooperative approach, Acta Appl. Math. Sinica*, 28 (2012), 1–20.

[29] J. Yong, and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.

7 Appendix

In this appendix, we present some detailed calculations.

Example 2.1. Recall that we are considering the following one-dimensional controlled linear SDE:

$$\begin{cases} dX(s) = u(s)ds + \sigma X(s)dW(s), & s \in [t,T], \\ X(t) = x, \end{cases} \tag{7.1}$$

with cost functional

$$J(t,x;u(\cdot)) = \mathbb{E}_t\Big[\int_t^T |u(s)|^2 ds + g(t)|X(T)|^2\Big], \tag{7.2}$$

where $\sigma > 0$ is a constant and $g(t)$ is a deterministic non-constant, continuous and positive function. For such a linear-quadratic optimal control problem on $[t,T]$ (with deterministic coefficients) and with $t \in [0,T)$ fixed, the Riccati equation takes the following form: (note that $t \in [0,T)$ is a parameter)

$$\begin{cases} P_s(s,t) = P(s,t)^2 - \sigma^2 P(s,t), & s \in [t,T], \\ P(T,t) = g(t). \end{cases}$$

Let us solve the above Riccati equation. By separation of variables, we have

$$ds = \frac{dP}{P^2 - \sigma^2 P} = \frac{1}{\sigma^2}\Big(\frac{1}{P-\sigma^2} - \frac{1}{P}\Big)dP.$$

Integrating from s to T, one has

$$\sigma^2(T-s) = \ln\Big(\frac{g(t)-\sigma^2}{g(t)}\Big) - \ln\Big(\frac{P(s,t)-\sigma^2}{P(s,t)}\Big) = \ln\Big(\frac{g(t)-\sigma^2}{g(t)}\,\frac{P(s,t)}{P(s,t)-\sigma^2}\Big).$$

Then

$$\frac{P(s,t)[g(t)-\sigma^2]}{[P(s,t)-\sigma^2]g(t)} = e^{\sigma^2(T-s)}.$$

Hence,

$$P(s,t) = \frac{\sigma^2 g(t)e^{\sigma^2(T-s)}}{\sigma^2 + g(t)(e^{\sigma^2(T-s)}-1)}, \qquad s \in [t,T],$$

and the optimal control is given by

$$\bar u(s) = -P(s,t)\bar X(s), \qquad s \in [t,T].$$

Thus, the closed-loop SDE reads

$$\begin{cases} d\bar{X}(s) = -P(s,t)\bar{X}(s)ds + \sigma\bar{X}(s)dW(s), \qquad s \in [t,T], \\ \bar{X}(t) = x. \end{cases}$$

Consequently, the optimal state process is given by

$$\begin{aligned} \bar{X}(s) &= xe^{-\int_t^s [P(\theta,t)+\frac{\sigma^2}{2}]d\theta + \sigma[W(s)-W(t)]} \\ &= \frac{\sigma^2 + g(t)(e^{\sigma^2(T-s)}-1)}{\sigma^2 + g(t)(e^{\sigma^2(T-t)}-1)}\, e^{-\frac{\sigma^2}{2}(s-t)+\sigma[W(s)-W(t)]}x, \qquad s \in [t,T]. \end{aligned}$$

since

$$\begin{aligned} -\int_t^s P(\theta,t)d\theta &= -\int_t^s \frac{\sigma^2 g(t)e^{\sigma^2(T-\theta)}}{\sigma^2 + g(t)(e^{\sigma^2(T-\theta)}-1)}\, d\theta \\ &= \int_t^s \frac{d[\sigma^2 + g(t)(e^{\sigma^2(T-\theta)}-1)]}{\sigma^2 + g(t)(e^{\sigma^2(T-\theta)}-1)} = \ln\Big(\frac{\sigma^2 + g(t)(e^{\sigma^2(T-s)}-1)}{\sigma^2 + g(t)(e^{\sigma^2(T-t)}-1)}\Big). \end{aligned}$$

Consequently, the optimal control also admits the following open-loop form:

$$\bar{u}(s) = \bar{u}(s;t,x) = -P(s,t)\bar{X}(s) = -\frac{\sigma^2 g(t)e^{\sigma^2(T-s)}e^{-\frac{\sigma^2}{2}(s-t)+\sigma[W(s)-W(t)]}x}{\sigma^2 + g(t)(e^{\sigma^2(T-t)}-1)}.$$

Recall that

$$\mathbb{E}_t\Big[e^{k[W(s)-W(t)]}\Big] = e^{\frac{k^2}{2}(s-t)}, \qquad s \ge t \ge 0.$$

In fact, for fixed t, applying Itô's formula to $s \mapsto \varphi(s) \equiv e^{k[W(s)-W(t)]}$, we have

$$\begin{cases} d\varphi(s) = k\varphi(s)dW(s) + \dfrac{k^2}{2}\varphi(s)ds, \qquad s \ge t, \\ \varphi(t) = 1. \end{cases}$$

Hence,

$$\begin{cases} d\{\mathbb{E}_t[\varphi(s)]\} = \dfrac{k^2}{2}\mathbb{E}_t[\varphi(s)], \qquad s \ge t, \\ \mathbb{E}_t[\varphi(t)] = 1, \end{cases}$$

which leads to what we want. We can check that

$$J(t,x;\bar{u}(\cdot)) = \mathbb{E}_t\Big[\int_t^T |\bar{u}(s)|^2 ds + g(t)|\bar{X}(T)|^2\Big] = \frac{\sigma^2 g(t)e^{\sigma^2(T-t)}x^2}{\sigma^2 + g(t)(e^{\sigma^2(T-t)}-1)} = P(t,t)x^2.$$

Next, let $\tau \in (t,T)$. We consider the corresponding LQ problem starting from the initial pair $(\tau, \bar{X}(\tau))$. Then the optimal state process, denoted by $\widehat{X}(\cdot)$, must be given by

$$\widehat{X}(s) = \frac{\sigma^2 + g(\tau)(e^{\sigma^2(T-s)}-1)}{\sigma^2 + g(\tau)(e^{\sigma^2(T-\tau)}-1)}\, e^{-\frac{\sigma^2}{2}(s-\tau)+\sigma[W(s)-W(\tau)]}\bar{X}(\tau), \qquad s \in [\tau,T],$$

and the optimal control, denoted by $\widehat{u}(\cdot)$, should be given by

$$\widehat{u}(s) = -P(s,\tau)\widehat{X}(s) = -\frac{\sigma^2 g(\tau)e^{\sigma^2(T-s)}e^{-\frac{\sigma^2}{2}(s-\tau)+\sigma[W(s)-W(\tau)]}\bar{X}(\tau)}{\sigma^2 + g(\tau)(e^{\sigma^2(T-\tau)}-1)}, \qquad s \in [\tau,T].$$

Note that

$$J(\tau,\bar{X}(\tau);\widehat{u}(\cdot)) = P(\tau,\tau)|\bar{X}(\tau)|^2 = \frac{\sigma^2 g(\tau)e^{\sigma^2(T-\tau)}}{\sigma^2 + g(\tau)(e^{\sigma^2(T-\tau)}-1)}|\bar{X}(\tau)|^2.$$

On the other hand,

$$\bar X(\tau)=\frac{\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)}{\sigma^2+g(t)(e^{\sigma^2(T-t)}-1)}\,e^{-\frac{\sigma^2}{2}(\tau-t)+\sigma[W(\tau)-W(t)]}x.$$

Thus, for $s\in[\tau,T]$,

$$\begin{aligned}\bar X(s)&=\frac{\sigma^2+g(t)(e^{\sigma^2(T-s)}-1)}{\sigma^2+g(t)(e^{\sigma^2(T-t)}-1)}\,e^{-\frac{\sigma^2}{2}(s-t)+\sigma[W(s)-W(t)]}x\\&=\frac{\sigma^2+g(t)(e^{\sigma^2(T-s)}-1)}{\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)}\,e^{-\frac{\sigma^2}{2}(s-\tau)+\sigma[W(s)-W(\tau)]}\bar X(\tau),\end{aligned}$$

and

$$\begin{aligned}&\bar u(s)=-P(s,t)\bar X(s)\\&=-\frac{\sigma^2 g(t)e^{\sigma^2(T-s)}}{\sigma^2+g(t)(e^{\sigma^2(T-s)}-1)}\cdot\frac{\sigma^2+g(t)(e^{\sigma^2(T-s)}-1)}{\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)}e^{-\frac{\sigma^2}{2}(s-\tau)+\sigma[W(s)-W(\tau)]}\bar X(\tau)\\&=-\frac{\sigma^2 g(t)e^{\sigma^2(T-s)}}{\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)}\,e^{-\frac{\sigma^2}{2}(s-\tau)+\sigma[W(s)-W(\tau)]}\bar X(\tau).\end{aligned}$$

Then

$$\begin{aligned}&J(\tau,\bar X(\tau);\bar u(\cdot))=\mathbb{E}_\tau\Big[\int_\tau^T|\bar u(s)|^2ds+g(\tau)|\bar X(T)|^2\Big]\\&=\mathbb{E}_\tau\Big[\int_\tau^T\frac{\sigma^4 g(t)^2e^{2\sigma^2(T-s)-\sigma^2(s-\tau)+2\sigma[W(s)-W(\tau)]}|\bar X(\tau)|^2}{[\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)]^2}\,ds\\&\qquad+\frac{\sigma^4 g(\tau)e^{-\sigma^2(T-\tau)+2\sigma[W(T)-W(\tau)]}|\bar X(\tau)|^2}{[\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)]^2}\Big]\\&=\frac{\sigma^4 g(t)^2|\bar X(\tau)|^2}{[\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)]^2}\int_\tau^T e^{2\sigma^2(T-s)+\sigma^2(s-\tau)}ds+\frac{\sigma^4 g(\tau)e^{\sigma^2(T-\tau)}|\bar X(\tau)|^2}{[\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)]^2}\\&=\frac{\sigma^2 g(t)^2|\bar X(\tau)|^2e^{\sigma^2(T-\tau)}(e^{\sigma^2(T-\tau)}-1)+\sigma^4 g(\tau)e^{\sigma^2(T-\tau)}|\bar X(\tau)|^2}{[\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)]^2}\\&=\frac{[g(t)^2(e^{\sigma^2(T-\tau)}-1)+\sigma^2 g(\tau)]\sigma^2e^{\sigma^2(T-\tau)}|\bar X(\tau)|^2}{[\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)]^2}.\end{aligned}$$

Consequently, the following holds:

$$\begin{aligned}&J(\tau,\bar X(\tau);\bar u(\cdot))-J(\tau,\bar X(\tau);\widehat u(\cdot))\\&=\Big\{\frac{g(t)^2(e^{\sigma^2(T-\tau)}-1)+\sigma^2 g(\tau)}{[\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)]^2}-\frac{g(\tau)}{\sigma^2+g(\tau)(e^{\sigma^2(T-\tau)}-1)}\Big\}\sigma^2e^{\sigma^2(T-\tau)}|\bar X(\tau)|^2\\&=\frac{\sigma^4(e^{\sigma^2(T-\tau)}-1)e^{\sigma^2(T-\tau)}[g(t)-g(\tau)]^2|\bar X(\tau)|^2}{[\sigma^2+g(t)(e^{\sigma^2(T-\tau)}-1)]^2[\sigma^2+g(\tau)(e^{\sigma^2(T-\tau)}-1)]}\\&=\frac{\sigma^4(e^{\sigma^2(T-\tau)}-1)[g(t)-g(\tau)]^2e^{\sigma^2(T-\tau)-\sigma^2(\tau-t)+2\sigma[W(\tau)-W(t)]}x^2}{[\sigma^2+g(t)(e^{\sigma^2(T-t)}-1)]^2[\sigma^2+g(\tau)(e^{\sigma^2(T-\tau)}-1)]},\end{aligned}$$

which is strictly positive unless $x=0$, or $g(\tau)=g(t)$. This means that the problem is time-inconsistent.

Example 2.2. (Generalized Merton's portfolio problem) Recall the following controlled SDE:

$$\left\{\begin{aligned}&dX(s)=\big[rX(s)+(\mu-r)u(s)-c(s)\big]ds+\sigma u(s)dW(s),\qquad s\in[t,T],\\&X(t)=x,\end{aligned}\right.\tag{7.3}$$

with payoff functional

$$J(t,x;u(\cdot),c(\cdot))=\mathbb{E}_t\Big[\int_t^T\nu(t,s)c(s)^\beta ds+\rho(t)X\big(T;t,x,u(\cdot),c(\cdot)\big)^\beta\Big],\tag{7.4}$$

where $\nu(\cdot\,,\cdot)$ and $\rho(\cdot)$ are given positive-valued functions, and $\beta \in (0,1)$. As a convention, we define

$$x^\beta = -\infty, \qquad x < 0.$$

The optimal control problem is to find a pair $(\bar u(\cdot), \bar c(\cdot))$ such that $J(t,x;u(\cdot),c(\cdot))$ is maximized. To approach this problem, we use dynamic programming method. More precisely, for any $(s,y) \in [t,T) \times [0,\infty)$, let

$$J^t\big(s,y;u(\cdot),c(\cdot)\big) = \mathbb{E}_t\Big[\int_s^T \nu(t,\tau)c(\tau)^\beta d\tau + \rho(t)X\big(T;s,y,u(\cdot),c(\cdot)\big)^\beta\Big]. \tag{7.5}$$

Define the value function $V^t(\cdot\,,\cdot)$ (which is parameterized by $t \in [0,T]$) by the following:

$$V^t(s,y) = \sup_{(u(\cdot),c(\cdot))} J^t\big(s,y;u(\cdot),c(\cdot)\big), \qquad (s,y) \in [t,T] \times [0,\infty). \tag{7.6}$$

Due to the problem being a maximization problem, the above convention forces $X(s)$ to stay nonnegative, in particular, the initial state $x \geq 0$ has to be assumed. In another word, $V^t(s,x)$ is only defined on $[0,T] \times [0,\infty)$. By Girsanov's theorem, we know that

$$\widetilde W(s) = \frac{\mu - r}{\sigma}s + \sigma W(s), \qquad s \geq 0$$

is a standard Brownian motion, with the natural filtration (augmented by all the $\mathbb{P}$-null sets) coincides with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$. Then (7.3) is equivalent to the following:

$$\begin{cases} dX(s) = \big[rX(s) - c(s)\big]ds + \sigma u(s)d\widetilde W(s), \qquad s \in [t,T], \\ X(t) = x, \end{cases} \tag{7.7}$$

Therefore, for any initial pair $(t,x) \in [0,T) \times [0,\infty)$, and a control $(u(\cdot),c(\cdot))$, the unique solution $X(\cdot) \equiv X(\cdot\,;t,x,u(\cdot),c(\cdot))$ of the above is given by the following:

$$X(s) = e^{r(s-t)}x - \int_t^s e^{r(s-\tau)}c(\tau)d\tau + \sigma\int_t^s e^{r(s-\tau)}u(\tau)d\widetilde W(\tau), \qquad s \in [t,T]. \tag{7.8}$$

If the initial wealth $x = 0$, to keep the wealth $X(\cdot)$ non-negative, we have to take $u(\cdot) = c(\cdot) = 0$, leading to

$$X(s) = 0, \qquad s \in [t,T].$$

This means that

$$V^t(t,0) = 0, \qquad t \in [0,T], \tag{7.9}$$

which gives the boundary condition for $V^t(\cdot\,,\cdot)$ on $x = 0$. Now, let us return to (7.3)–(7.4). In the case that $V^t(\cdot\,,\cdot)$ is differentiable, it satisfies the following HJB equation:

$$\begin{aligned} 0 &= V^t_s(s,y) + \sup_{(u,c)}\Big[V^t_y(s,y)\big[ry + (\mu - r)u - c\big] + \frac{1}{2}\sigma^2u^2V^t_{yy}(s,y) + \nu(t,s)c^\beta\Big] \\ &= V^t_s(s,y) + ryV^t_y(s,y) + \sup_{u\in\mathbb{R}}\Big[(\mu - r)V^t_y(s,y)u + \frac{1}{2}\sigma^2V^t_{yy}(s,y)u^2\Big] + \sup_{c\geq 0}\big[\nu(t,s)c^\beta - cV^t_y(s,y)\big]. \end{aligned} \tag{7.10}$$

Assume, for the time being, that the following holds:

$$V^t_y(s,y) > 0, \quad V^t_{yy}(s,y) < 0, \qquad (s,y) \in [t,T] \times (0,\infty). \tag{7.11}$$

Then

$$\sup_{u\in\mathbb{R}}\Big[(\mu - r)V^t_y(s,y)u + \frac{1}{2}\sigma^2V^t_{yy}(s,y)u^2\Big] = -\frac{(\mu - r)^2V^t_y(s,y)^2}{2\sigma^2V^t_{yy}(s,y)} > 0, \tag{7.12}$$

with the maximum attained at

$$\bar{u}^t(s,y) = -\frac{(\mu - r)V_y^t(s,y)}{\sigma^2 V_{yy}^t(s,y)} > 0, \tag{7.13}$$

and

$$\sup_{c>0}\big[\nu(t,s)c^\beta - cV_y^t(s,y)\big] = (1-\beta)\beta^{\frac{\beta}{1-\beta}}\nu(t,s)^{\frac{1}{1-\beta}}V_y^t(s,y)^{\frac{\beta}{\beta-1}} > 0, \tag{7.14}$$

with the maximum attained at

$$\bar{c}^t(s,y) = \Big(\frac{\beta\nu(t,s)}{V_y^t(s,y)}\Big)^{\frac{1}{1-\beta}} > 0. \tag{7.15}$$

Consequently, the HJB equation reads

$$\begin{cases} V_s^t(s,y) + ryV_y^t(s,y) - \dfrac{(\mu-r)^2 V_y^t(s,y)^2}{2\sigma^2 V_{yy}^t(s,y)} + (1-\beta)\beta^{\frac{\beta}{1-\beta}}\nu(t,s)^{\frac{1}{1-\beta}}V_y^t(s,y)^{\frac{\beta}{\beta-1}} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (s,y) \in [t,T)\times(0,\infty), \\ V^t(s,0) = 0, \qquad\qquad s \in [t,T], \\ V^t(T,y) = \rho(t)y^\beta, \qquad\qquad y \in (0,\infty). \end{cases} \tag{7.16}$$

We try to find the solution of the following form:

$$V^t(s,y) = \varphi(s)y^\beta, \qquad (s,y) \in [t,T]\times[0,\infty). \tag{7.17}$$

Clearly, with such a form, (7.11) is satisfied. Then we should have

$$0 = \varphi'(s)y^\beta + r\beta\varphi(s)y^\beta + \frac{(\mu-r)^2\beta\varphi(s)y^\beta}{2\sigma^2(1-\beta)} + (1-\beta)\nu(t,s)^{\frac{1}{1-\beta}}\varphi(s)^{\frac{\beta}{\beta-1}}y^\beta$$

This leads to an ordinary differential equation for $\varphi(\cdot)$:

$$\begin{cases} \varphi'(s) + \Big[r\beta + \dfrac{(\mu-r)^2\beta}{2\sigma^2(1-\beta)}\Big]\varphi(s) + (1-\beta)\nu(t,s)^{\frac{1}{1-\beta}}\varphi(s)^{\frac{\beta}{\beta-1}} = 0, \quad s\in[t,T], \\ \varphi(T) = \rho(t). \end{cases} \tag{7.18}$$

If we denote

$$\lambda = r\beta + \frac{(\mu-r)^2\beta}{2\sigma^2(1-\beta)} = \frac{[2r\sigma^2(1-\beta) + (\mu-r)^2]\beta}{2\sigma^2(1-\beta)}, \tag{7.19}$$

then (7.18) becomes

$$\begin{cases} \varphi'(s) + \lambda\varphi(s) + (1-\beta)\nu(t,s)^{\frac{1}{1-\beta}}\varphi(s)^{\frac{\beta}{\beta-1}} = 0, \qquad s\in[t,T], \\ \varphi(T) = \rho(t). \end{cases} \tag{7.20}$$

This is a Bernoulli equation. To solve it, let

$$\psi(s) = \varphi(s)^{\frac{1}{1-\beta}}, \qquad s \in [t,T]. \tag{7.21}$$

Then

$$\begin{aligned} \psi'(s) &= \frac{1}{1-\beta}\varphi(s)^{\frac{\beta}{1-\beta}}\varphi'(s) = -\frac{1}{1-\beta}\varphi(s)^{\frac{\beta}{1-\beta}}\Big(\lambda\varphi(s) + (1-\beta)\nu(t,s)^{\frac{1}{1-\beta}}\varphi(s)^{\frac{\beta}{\beta-1}}\Big) \\ &= -\frac{\lambda}{1-\beta}\psi(s) - \nu(t,s)^{\frac{1}{1-\beta}}, \qquad s\in[t,T], \end{aligned} \tag{7.22}$$

with

$$\psi(T) = \rho(t)^{\frac{1}{1-\beta}}. \tag{7.23}$$

Hence,

$$\psi(s)=e^{\frac{\lambda}{1-\beta}(T-s)}\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{\frac{\lambda}{1-\beta}(\tau-s)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau,\qquad s\in[t,T]. \tag{7.24}$$

Then

$$\varphi(s)=\Big[e^{\frac{\lambda}{1-\beta}(T-s)}\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{\frac{\lambda}{1-\beta}(\tau-s)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau\Big]^{1-\beta},\qquad s\in[t,T]. \tag{7.25}$$

Consequently,

$$V^t(s,y)=\Big[e^{\frac{\lambda}{1-\beta}(T-s)}\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{\frac{\lambda}{1-\beta}(\tau-s)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau\Big]^{1-\beta}y^\beta,\qquad (s,y)\in[t,T]\times[0,\infty). \tag{7.26}$$

Therefore, the optimal control $(\bar u^t(\cdot),\bar c^t(\cdot))$ for the initial pair (t,x) is given by the following:

$$\bar u^t(s)=-\frac{(\mu-r)V_y^t(s,\bar X^t(s))}{\sigma^2[V_{yy}^t(s,\bar X^t(s))]}=\frac{(\mu-r)\bar X^t(s)}{\sigma^2(1-\beta)},\qquad s\in[t,T], \tag{7.27}$$

which is not depending on the parameter t directly, and

$$\begin{aligned}\bar c^t(s)&=\beta^{\frac{1}{1-\beta}}\nu(t,s)^{\frac{1}{1-\beta}}V_y^t(s,\bar X^t(s))^{\frac{1}{\beta-1}}=\nu(t,s)^{\frac{1}{1-\beta}}\varphi(s)^{\frac{1}{\beta-1}}\bar X^t(s)\\&=\frac{\nu(t,s)^{\frac{1}{1-\beta}}\bar X^t(s)}{e^{\frac{\lambda}{1-\beta}(T-s)}\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{\frac{\lambda}{1-\beta}(\tau-s)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau},\qquad s\in[t,T],\end{aligned} \tag{7.28}$$

which is depending on the parameter t directly. Combining the above, we have

$$\begin{aligned}&\sup_{u(\cdot),c(\cdot)}J(t,x;u(\cdot),c(\cdot))=V(t,x)\equiv V^t(t,x)\\&\quad=\Big[e^{\frac{\lambda}{1-\beta}(T-t)}\rho(t)^{\frac{1}{1-\beta}}+\int_t^T e^{\frac{\lambda}{1-\beta}(\tau-t)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau\Big]^{1-\beta}x^\beta\\&\quad=\mathbb{E}_t\Big[\int_t^T\nu(t,\tau)\bar c^t(\tau)^\beta d\tau+\rho(t)\bar X^t(T)^\beta\Big],\qquad (t,x)\in[0,T]\times[0,\infty).\end{aligned} \tag{7.29}$$

Now, let $\bar t\in(t,T)$. Then, applying the above argument, one has

$$V(\bar t,\bar X^t(\bar t))=\Big[e^{\frac{\lambda}{1-\beta}(T-\bar t)}\rho(\bar t)^{\frac{1}{1-\beta}}+\int_{\bar t}^T e^{\frac{\lambda}{1-\beta}(\tau-\bar t)}\nu(\bar t,\tau)^{\frac{1}{1-\beta}}d\tau\Big]^{1-\beta}\bar X^t(\bar t)^\beta. \tag{7.30}$$

We claim that if the following is assumed

$$\int_{\bar t}^T\Big[\frac{e^{\lambda\tau}\nu(t,\tau)}{\rho(t)}\Big]^{\frac{1}{1-\beta}}d\tau\neq\int_{\bar t}^T\Big[\frac{e^{\lambda\tau}\nu(\bar t,\tau)}{\rho(\bar t)}\Big]^{\frac{1}{1-\beta}}d\tau, \tag{7.31}$$

then

$$J\big(\bar t,\bar X^t(\bar t);\bar u(\cdot)\big|_{[\bar t,T]},\bar c(\cdot)\big|_{[\bar t,T]}\big)<V(\bar t,\bar X^t(\bar t)). \tag{7.32}$$

This means that the restriction $\big(\bar u(\cdot)\big|_{[\bar t,T]},\bar c(\cdot)\big|_{[\bar t,T]}\big)$ of $(\bar u(\cdot),\bar c(\cdot))$ on $[\bar t,T]$ is not optimal for the initial pair $(\bar t,\bar X^t(\bar t))$. To prove our claim, let us denote

$$\begin{aligned}\Gamma(t,s)&=\frac{\nu(t,s)^{\frac{1}{1-\beta}}}{e^{\frac{\lambda}{1-\beta}(T-s)}\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{\frac{\lambda}{1-\beta}(\tau-s)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau}\\&=\frac{e^{-\frac{\lambda}{1-\beta}(T-s)}\nu(t,s)^{\frac{1}{1-\beta}}}{\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{-\frac{\lambda}{1-\beta}(T-\tau)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau}\end{aligned} \tag{7.33}$$

Then

$$\bar c^t(s)=\Gamma(t,s)\bar X^t(s),\qquad \sigma\bar u^t(s)=\frac{\mu-r}{\sigma(1-\beta)}\bar X^t(s),\qquad s\in[t,T],$$

where $\bar X^t(\cdot)$ is the solution to the following closed-loop system

$$\begin{cases} d\bar X^t(s)=\Big(r+\dfrac{(\mu-r)^2}{\sigma^2(1-\beta)}-\Gamma(t,s)\Big)\bar X^t(s)ds+\dfrac{\mu-r}{\sigma(1-\beta)}\bar X^t(s)dW(s), \qquad s\in[t,T],\\ \bar X^t(t)=x.\end{cases}$$

By denoting

$$b=r+\frac{(\mu-r)^2}{\sigma^2(1-\beta)}\,,\qquad a=\frac{\mu-r}{\sigma(1-\beta)}\,,$$

we may write the above as

$$\begin{cases} d\bar X^t(s)=\big[b-\Gamma(t,s)\big]\bar X^t(s)ds+a\bar X^t(s)dW(s),\qquad s\in[t,T],\\ \bar X^t(t)=x.\end{cases}$$

Hence,

$$\bar X^t(s)=\bar X^t(\bar t)e^{(b-\frac{a^2}{2})(s-\bar t)-\int_{\bar t}^s\Gamma(t,\tau)d\tau+a[W(s)-W(\bar t)]},\qquad s\in[\bar t,T].$$

Consequently,

$$\begin{aligned}\mathbb{E}_{\bar t}\big[\bar X^t(s)^\beta\big]&=\bar X^t(\bar t)^\beta e^{\beta(b-\frac{a^2}{2})(s-\bar t)-\beta\int_{\bar t}^s\Gamma(t,\tau)d\tau+\frac{\beta^2a^2}{2}(s-\bar t)}\\ &=\bar X^t(\bar t)^\beta e^{\lambda(s-\bar t)-\beta\int_{\bar t}^s\Gamma(t,\tau)d\tau},\qquad s\in[\bar t,T].\end{aligned}$$

Here, we note that (recall (7.19))

$$\beta\Big(b-\frac{a^2}{2}\Big)+\frac{\beta^2a^2}{2}=\beta\Big(b-\frac{a^2(1-\beta)}{2}\Big)=\beta\Big(r+\frac{(\mu-r)^2}{2\sigma^2(1-\beta)}\Big)=\lambda.$$

Then

$$\begin{aligned}&J(\bar t,\bar X^t(\bar t);\bar u(\cdot)\big|_{[\bar t,T]},\bar c(\cdot)\big|_{[\bar t,T]})=\mathbb{E}_{\bar t}\Big[\int_{\bar t}^T\nu(\bar t,s)\bar c^t(s)^\beta ds+\rho(\bar t)\bar X^t(T)^\beta\Big]\\ &=\int_{\bar t}^T\nu(\bar t,s)\Gamma(t,s)^\beta\mathbb{E}_{\bar t}[\bar X^t(s)^\beta]ds+\rho(\bar t)\mathbb{E}_{\bar t}[\bar X^t(T)^\beta]\\ &=\Big[\int_{\bar t}^T\nu(\bar t,s)\Gamma(t,s)^\beta e^{\lambda(s-\bar t)-\beta\int_{\bar t}^s\Gamma(t,\tau)d\tau}ds+\rho(\bar t)e^{\lambda(T-\bar t)-\beta\int_{\bar t}^T\Gamma(t,\tau)d\tau}\Big]\bar X^t(\bar t)^\beta\\ &=\Big[\int_{\bar t}^Te^{\lambda(s-\bar t)}\nu(\bar t,s)\Big(\Gamma(t,s)e^{-\int_{\bar t}^s\Gamma(t,\tau)d\tau}\Big)^\beta ds+e^{\lambda(T-\bar t)}\rho(\bar t)e^{-\beta\int_{\bar t}^T\Gamma(t,\tau)d\tau}\Big]\bar X^t(\bar t)^\beta.\end{aligned}$$

By Hölder's inequality, one has

$$\begin{aligned}&\int_{\bar t}^Te^{\lambda(s-\bar t)}\nu(\bar t,s)\Big(\Gamma(t,s)e^{-\int_{\bar t}^s\Gamma(t,\tau)d\tau}\Big)^\beta ds+e^{\lambda(T-\bar t)}\rho(\bar t)e^{-\beta\int_{\bar t}^T\Gamma(t,\tau)d\tau}\\ &\le\Big[\int_{\bar t}^Te^{\frac{\lambda}{1-\beta}(s-\bar t)}\nu(\bar t,s)^{\frac{1}{1-\beta}}ds\Big]^{1-\beta}\Big[\int_{\bar t}^T\Gamma(t,s)e^{-\int_{\bar t}^s\Gamma(t,\tau)d\tau}ds\Big]^\beta+e^{\lambda(T-\bar t)}\rho(\bar t)e^{-\beta\int_{\bar t}^T\Gamma(t,\tau)d\tau}\\ &\le\Big[\int_{\bar t}^Te^{\frac{\lambda}{1-\beta}(s-\bar t)}\nu(\bar t,s)^{\frac{1}{1-\beta}}ds\Big]^{1-\beta}\Big[1-e^{-\int_{\bar t}^T\Gamma(t,\tau)d\tau}\Big]^\beta+e^{\lambda(T-\bar t)}\rho(\bar t)e^{-\beta\int_{\bar t}^T\Gamma(t,\tau)d\tau}\\ &\equiv\alpha_1(\bar t)^{1-\beta}\gamma(t,\bar t)^\beta+\alpha_2(\bar t)^{1-\beta}\big[1-\gamma(t,\bar t)\big]^\beta,\end{aligned}$$

where

$$\begin{cases}\alpha_1(\bar t)=\displaystyle\int_{\bar t}^Te^{\frac{\lambda}{1-\beta}(s-\bar t)}\nu(\bar t,s)^{\frac{1}{1-\beta}}ds,\qquad \alpha_2(\bar t)=e^{\frac{\lambda}{1-\beta}(T-\bar t)}\rho(\bar t)^{\frac{1}{1-\beta}}\ge0,\\ \gamma(t,\bar t)=1-e^{-\int_{\bar t}^T\Gamma(t,\tau)d\tau}\in[0,1].\end{cases}$$

To proceed further, we need the following elementary result.

Lemma 7.1. Let $\alpha_1, \alpha_2 > 0$, $\beta \in (0,1)$, and

$$f(\gamma) = \alpha_1^{1-\beta}\gamma^\beta + \alpha_2^{1-\beta}(1-\gamma)^\beta, \qquad \gamma \in [0,1].$$

Then $\gamma \mapsto f(\gamma)$ is strictly concave and

$$\max_{\gamma\in[0,1]} f(\gamma) = f\Big(\frac{\alpha_1}{\alpha_1+\alpha_2}\Big) = (\alpha_1+\alpha_2)^{1-\beta}.$$

Proof. Note that

$$f'(\gamma) = \beta\Big[\alpha_1^{1-\beta}\gamma^{\beta-1} - \alpha_2^{1-\beta}(1-\gamma)^{\beta-1}\Big],$$

and

$$f''(\gamma) = -\beta(1-\beta)\Big[\alpha_1^{1-\beta}\gamma^{\beta-2} + \alpha_2^{1-\beta}(1-\gamma)^{\beta-2}\Big] < 0, \qquad \forall \gamma \in (0,1).$$

Thus, $\gamma \mapsto f(\gamma)$ is strictly concave. By setting $f'(\gamma) = 0$, we have

$$\frac{\alpha_1}{\gamma} = \frac{\alpha_2}{1-\gamma}\,,$$

which gives the unique maximum:

$$\gamma = \frac{\alpha_1}{\alpha_1+\alpha_2}\,.$$

Clearly,

$$f\Big(\frac{\alpha_1}{\alpha_1+\alpha_2}\Big) = \alpha_1^{1-\beta}\frac{\alpha_1^\beta}{(\alpha_1+\alpha_2)^\beta} + \alpha_2^{1-\beta}\frac{\alpha_2^\beta}{(\alpha_1+\alpha_2)^\beta} = (\alpha_1+\alpha_2)^{1-\beta}.$$

This proves the lemma. $\square$

By the above lemmas, we obtain (note (7.30))

$$\begin{aligned}
&J\big(\bar t, \bar X^t(\bar t); \bar u(\cdot)\big|_{[\bar t,T]}, \bar c(\cdot)\big|_{[\bar t,T]}\big)\\
&= \Big[\int_{\bar t}^T e^{\lambda(s-\bar t)}\nu(\bar t,s)\Big(\Gamma(t,s)e^{-\int_{\bar t}^s \Gamma(t,\tau)d\tau}\Big)^\beta ds + e^{\lambda(T-\bar t)}\rho(\bar t)e^{-\beta\int_{\bar t}^T \Gamma(t,\tau)d\tau}\Big]\bar X^t(\bar t)^\beta\\
&\le \Big[\alpha_1(\bar t)^{1-\beta}\gamma(t,\bar t)^\beta + \alpha_2(\bar t)^{1-\beta}[1-\gamma(t,\bar t)]^\beta\Big]\bar X^t(\bar t)^\beta \le [\alpha_1(\bar t)+\alpha_2(\bar t)]^{1-\beta}\bar X^t(\bar t)^\beta\\
&= \Big[\int_{\bar t}^T e^{\frac{\lambda}{1-\beta}(s-\bar t)}\nu(\bar t,s)^{\frac{1}{1-\beta}}ds + e^{\frac{\lambda}{1-\beta}(T-\bar t)}\rho(\bar t)^{\frac{1}{1-\beta}}\Big]^{1-\beta}\bar X^t(\bar t)^\beta = V(\bar t, \bar X^t(\bar t)).
\end{aligned}$$

In order to have a strict inequality in the above, it suffices to have

$$\alpha_1(\bar t)^{1-\beta}\gamma(t,\bar t)^\beta + \alpha_2(\bar t)^{1-\beta}[1-\gamma(t,\bar t)]^\beta < [\alpha_1(\bar t)+\alpha_2(\bar t)]^{1-\beta},$$

which is implied by

$$\gamma(t,\bar t) \ne \frac{\alpha_1(\bar t)}{\alpha_1(\bar t)+\alpha_2(\bar t)}\,.$$

This is equivalent to the following:

$$\begin{aligned}
&e^{-\int_{\bar t}^T \Gamma(t,\tau)d\tau} \ne \frac{\alpha_2(\bar t)}{\alpha_1(\bar t)+\alpha_2(\bar t)} = \frac{e^{\frac{\lambda}{1-\beta}(T-\bar t)}\rho(\bar t)^{\frac{1}{1-\beta}}}{e^{\frac{\lambda}{1-\beta}(T-\bar t)}\rho(\bar t)^{\frac{1}{1-\beta}} + \int_{\bar t}^T e^{\frac{\lambda}{1-\beta}(s-\bar t)}\nu(\bar t,s)^{\frac{1}{1-\beta}}ds}\\
&= \frac{\rho(\bar t)^{\frac{1}{1-\beta}}}{\rho(\bar t)^{\frac{1}{1-\beta}} + \int_{\bar t}^T e^{-\frac{\lambda}{1-\beta}(T-s)}\nu(\bar t,s)^{\frac{1}{1-\beta}}ds}.
\end{aligned}$$

Note that (recall (7.33))

$$
\begin{aligned}
&-\int_{\bar t}^{T}\Gamma(t,s)ds=-\int_{\bar t}^{T}\frac{e^{-\frac{\lambda}{1-\beta}(T-s)}\nu(t,s)^{\frac{1}{1-\beta}}}{\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{-\frac{\lambda}{1-\beta}(T-\tau)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau}\,ds\\
&=\int_{\bar t}^{T}\frac{d[\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{-\frac{\lambda}{1-\beta}(T-\tau)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau]}{\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{-\frac{\lambda}{1-\beta}(T-\tau)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau}\\
&=\ln\Big[\rho(t)^{\frac{1}{1-\beta}}+\int_s^T e^{-\frac{\lambda}{1-\beta}(T-\tau)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau\Big]\Big|_{\bar t}^{T}=\ln\frac{\rho(t)^{\frac{1}{1-\beta}}}{\rho(t)^{\frac{1}{1-\beta}}+\int_{\bar t}^T e^{-\frac{\lambda}{1-\beta}(T-\tau)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau}.
\end{aligned}
$$

Hence, we need

$$
\frac{\rho(t)^{\frac{1}{1-\beta}}}{\rho(t)^{\frac{1}{1-\beta}}+\int_{\bar t}^T e^{-\frac{\lambda}{1-\beta}(T-\tau)}\nu(t,\tau)^{\frac{1}{1-\beta}}d\tau}\neq\frac{\rho(\bar t)^{\frac{1}{1-\beta}}}{\rho(\bar t)^{\frac{1}{1-\beta}}+\int_{\bar t}^T e^{-\frac{\lambda}{1-\beta}(T-\tau)}\nu(\bar t,\tau)^{\frac{1}{1-\beta}}d\tau}.
$$

which is equivalent to (7.31), proving our claim.

Note that in the case

$$
\nu(t,s)=e^{-\delta(s-t)},\qquad\rho(t)=e^{-\delta(T-t)},\qquad 0\le t\le s\le T, \tag{7.34}
$$

we have

$$
\frac{\nu(t,\tau)}{\rho(t)}=\frac{e^{-\delta(\tau-t)}}{e^{-\delta(T-t)}}=e^{\delta(T-\tau)}.
$$

Thus, (7.31) will not be true. When (7.34) holds, the problem is referred to as the (classical) Merton's portfolio problem. In this case, (with $a=\frac{\lambda-\delta}{1-\beta}$)

$$
\begin{aligned}
\bar c(s,y)&=\frac{e^{-\frac{\delta}{1-\beta}(s-t)}y}{e^{\kappa(T-s)}e^{-\frac{\delta}{1-\beta}(T-t)}+\int_s^T e^{\kappa(\tau-s)}e^{-\frac{\delta}{1-\beta}(\tau-t)}d\tau}\\
&=\frac{y}{e^{\kappa(T-s)}e^{-\frac{\delta(T-s)}{1-\beta}}+\int_s^T e^{\kappa(\tau-s)}e^{-\frac{\delta(\tau-s)}{1-\beta}}d\tau}\\
&=\frac{y}{e^{a(T-s)}+\int_s^T e^{a(\tau-s)}d\tau}=\frac{ay}{ae^{a(T-s)}+e^{a(T-s)}-1},
\end{aligned} \tag{7.35}
$$

which is independent of t. This recovers the solution to the classical Merton's portfolio problem.